1. Determine whether the following series converge or diverge with proof.

a. $\sum_{k=2}^{\infty} \sqrt{k^2+1}/\sqrt{k^5+k-2}$

b. $\sum_{k=1}^{\infty} (\sqrt{k}-\sqrt{k-1})^k$

c. $\sum_{n=1}^{\infty} n \sin(1/n)$

d. $\sum_{k=1}^{\infty} \dfrac{\sqrt[3]{k}-1}{k(\sqrt{k}+1)}$

e. $\sum_{k=2}^{\infty} k/[(1+k^2)\ln(1+k^2)]$

f. $\sum_{k=1}^{\infty} \sqrt{k!}/k^k$

g. $\sum_{k=1}^{\infty} \dfrac{k \ln k}{(k+1)^3}$

h. $\sum_{k=1}^{\infty} 1/(k^2+3k+2)$

Couldn't get these two

1 a) $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^5+k-2}}$

WTS CON

<u>Proof</u>

① Consider: $b_k = \frac{\sqrt{k^2}}{\sqrt{k^5}}$

$= \frac{k}{k^2\sqrt{k}}$

$= \frac{1}{k^{3/2}}$

② Now the series of $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

converges because it is a p-series with a $p = \frac{3}{2}$. Remember when $p > 1$ the series will converge.

③ So let a_k be the given series:

$$\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^5+k-2}}$$

Lets try the limit test:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\frac{\sqrt{k^2+1}}{\sqrt{k^5+k-2}}}{\frac{1}{k^{3/2}}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^5+k-2}} \cdot k^{3/2}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k^5+k^3}}{\sqrt{k^5+k-2}}$$

$$= \lim_{k \to \infty} \frac{\sqrt{k^5 + k^3}}{\sqrt{k^5 + k - 2}} \left(\frac{\frac{1}{\sqrt{k^5}}}{\frac{1}{\sqrt{k^5}}} \right)$$

$$= \lim_{k \to \infty} \frac{\sqrt{1 + \cancel{\frac{1}{k^2}}^{\to 0}}}{\sqrt{1 + \cancel{\frac{1}{k^4}}^{\to 0} - \cancel{\frac{2}{k^5}}^{\to 0}}}$$

$$= 1$$

(4) So, since the limit exists, and we know b_n convergos, by the limit test we know the series $\sum_{k=1}^{\infty} \frac{\sqrt{k^2+1}}{\sqrt{k^5+k-2}}$ convergos.

1b) WTS $\sum_{k=1}^{\infty} (\sqrt{k} - \sqrt{k-1})^k$ DIV

Proof

If the terms of a series a_n are not approaching 0 as $n \to \infty$, then the series diverges.

so: $\lim_{k \to \infty} (\sqrt{k} - \sqrt{k-1})^k = \infty$

so by the Divergence test the series diverges.

1c) WTS $\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$ DIV by divergence test.

Proof

If the terms of a series a_n are not approaching 0 as $n \to \infty$, then the series diverges.

so: $\lim_{n \to \infty} n \sin(\frac{1}{n})$

$= \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$ ← Indeterminate form "$\frac{0}{0}$"

L'H $= \lim_{n \to \infty} \frac{-\frac{\cos \frac{1}{n}}{n^2}}{-\frac{1}{n^2}}$

$= \lim_{n \to \infty} \cos(\frac{1}{n})^{0}$

$= \cos 0$

$\boxed{= 1}$ ∴ The series $\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$ diverges by failing the divergence test.

1e) WTS $\sum_{k=2}^{\infty} \frac{k}{(1+k^2)\ln(1+k^2)}$ DIV by Integral test:

Proof

If the Integral reprenstation of a series divergers, then the series also diverges.

① Consider:

$$f(x) = \frac{x}{(1+x^2)\ln(1+x^2)}$$

Lets check to see if the integral diverges:

$$\int_1^{\infty} \frac{x}{(1+x^2)\ln(1+x^2)}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{x}{(1+x^2)\ln(1+x^2)}\,dx$$

$$u = 1+x^2$$
$$du = 2x\,dx$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{1}{u\ln u}\,du$$

$$w = \ln u$$
$$dw = \frac{1}{u}\,dw$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_1^b \frac{1}{w}\,dw$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(\ln(1+x^2))\Big]_1^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\Big[\ln(\ln(1+b^2)) - \ln(\ln(1+1^2))\Big]$$

$$= \infty$$

② $\therefore$ By the Integral test, since the integral represtation of $\sum_{k=2}^{\infty} \frac{k}{(1+k^2)\ln(1+k^2)}$ divergers, the series also diverges.

4f) WTS $\sum_{k=1}^{\infty} \frac{\sqrt{k!}}{k^k}$ CON by root test

Proof

If

$$\lim_{n\to\infty} \sqrt[n]{a_n} > 1$$

then the series a_n absolutely converges.

Lets try the root test:

① $$\lim_{k\to\infty} \sqrt[k]{\frac{\sqrt{k!}}{k^k}}$$

$$= \lim_{k\to\infty} \frac{\sqrt[k]{\sqrt{k!}}}{k}$$

$$= \lim_{k\to\infty} \frac{\left(\sqrt{k!}\right)^{1/k}}{k}$$

$$= \lim_{k\to\infty} \frac{\left((k!)^{1/2}\right)^{1/k}}{k}$$

$$= \lim_{k\to\infty} \frac{(k!)^{\overset{\to 0}{1/2k}}}{k}$$

Note:
$$\lim_{k\to\infty} \frac{1}{2k} = 0$$

$$= \lim_{k\to\infty} \frac{1}{k}$$

$$= 0$$

Since the limit is less than 1, the series absolutely convergos. Since the series absolutely convergos, the series convergos.

g) WTS $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3}$ CON via comparison:

Proof

If I can choose a series b_k and $\sum_{k=1}^{\infty} \frac{k \ln k}{(k+1)^3} < b_k$, and b_k converges, then I can say the original given series converges.

① Consider:

$$b_k = \sum_{k=1}^{\infty} \frac{k \ln k}{k^3}$$
$$= \sum_{k=1}^{\infty} \frac{\ln k}{k^2}$$

② I can show convergence of b_k via integral test:

$$f(x) = \frac{\ln x}{x^2}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^2}\,dx$$

$u = \ln x$ $\quad dv = \frac{1}{x^2}$

$du = \frac{1}{x}dx$ $\quad v = -\frac{1}{x}$

$$= \lim_{b\to\infty} \left[\ln x \left(-\frac{1}{x}\right) - \int_1^b -\frac{1}{x}\frac{1}{x}\,dx \right]$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{x} + \int_1^b \frac{1}{x^2}\,dx \right]$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{x} + \left[-\frac{1}{x}\right]_1^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{\ln b}{b} - \cancel{\frac{1}{b}}^{\,0} - \left(-\cancel{\frac{\ln 1}{1}}^{\,0} - 1\right) \right]$$

$$= \lim_{b\to\infty} \left[-\cancel{\frac{\ln b}{b}}^{\,0} + 1 \right]$$

$$= 1$$

Note: $\lim_{b\to\infty}\left(-\frac{\ln b}{b}\right) = 0$ via L'Hospitals

③ Since the integral representation of b_k converges, the series b_k converges.

④ So lets make a comparison:

$$0 < a_k < b_k$$

$$0 < \frac{k \ln k}{(k+1)^3} < \frac{k \ln k}{k^3}$$

$\therefore$ Since the upper series converges, the lower series a_k also converges by the comparison test.

1n) WTS $\sum_{k=1}^{\infty} \frac{1}{k^2+3k+2}$ CON

Lets look at some terms:

$$= \frac{1}{1^2+3\cdot 1+2} + \frac{1}{2^2+3\cdot 2+2} + \frac{1}{3^2+3\cdot 3+2} + \cdots + \frac{1}{(k-1)^2+3(k-1)+2} + \frac{1}{k^2+3k+2}$$

$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+2)(k+1)}$$

2. Determine of the following series converges conditionally or absolutely.

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$$

Alternating

WTS $\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}$ CON using the ratio test

Proof

Lets setup the ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$$

If this limit is less than 1, the series will converge absolutely.

① $$\sum_{n=1}^{\infty} a_{n+1} = \sum_{n=1}^{\infty} \frac{2^{(n+1)}(n+1)!}{5 \cdot 8 \cdot 11 \cdots (3n+2) \cdot (3(n+1)+2)}$$

② $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left| \frac{\cancel{2^{n}}2\,(n+1)\cancel{!}\;\cancel{(-1)^{n}}}{\cancel{5 \cdot 8 \cdot 11 \cdot (3n+2)} \cdot (3(n+1)+2)} \cdot \frac{\cancel{5 \cdot 8 \cdot 11 \cdots (3n+2)}}{\cancel{2^n n!}\;\cancel{(-1)^{n+1}}} \right|$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{3(n+1)+2}$$

$$= \lim_{n\to\infty} \frac{2n+2}{3n+5} \left(\frac{\frac{1}{n}}{\frac{1}{n}} \right)$$

$$= \lim_{n\to\infty} \frac{2 + \cancelto{0}{\frac{2}{n}}}{3 + \cancelto{0}{\frac{5}{n}}}$$

$$= \frac{2}{3}$$

$\therefore$ Since the limit of the ratio is less than 1, the series converges absolutely. So the series converges.

3. Determine of the following series converges conditionally or absolutely.

$$\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{1 + n\sqrt{n}}$$

WTS $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{6}\right)}{1 + n\sqrt{n}}$

Proof

If Σa_n and $\Sigma |a_n|$ converge, then the series a_n converges absolutely.

So lets check:

① $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin\left(\frac{n\pi}{6}\right)}{1+n\sqrt{n}} \right|$

Remember: $-1 < \sin n < 1$

$|\sin n| < 1$

so: $|\sin \frac{n\pi}{6}| < 1$

$\left| \frac{\sin \frac{n\pi}{6}}{1+n\sqrt{n}} \right| < \frac{1}{1+n\sqrt{n}}$

② So lets see if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}}$ converges:

$$\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

I propose that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ is less than the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}}$

So, $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

This is a p-series with $p = \frac{3}{2}$.

since $p > 1$ the series converges.

③ So since $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges and $\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}}$ is less, it also converges.

④ Because $\left| \frac{\sin \frac{n\pi}{6}}{1+n\sqrt{n}} \right|$ is less than $\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}}$, and $\sum_{n=1}^{\infty} \frac{1}{1+n\sqrt{n}}$ converges, $\sum_{n=1}^{\infty} \left| \frac{\sin \frac{n\pi}{6}}{1+n\sqrt{n}} \right|$ also converges. By the comparison test the given series absolutely converges.

4. Find McLaurin series for the function f defined by $f(x) = \dfrac{1}{(1-x)^4}$ using two different techniques.

$C = 0$

$$f(x) = \frac{1}{(1-x)^4} = (1-x)^{-4}$$

$$f'(x) = -4(1-x)^{-4-1}(-1)$$

$$f''(x) = -4(-4-1)(1-x)^{-4-2}(-1)$$

$$f'''(x) = -4(-4-1)(-4-2)(1-x)^{-4-3}(-1)$$

$$f^n(x) = -4(-4-1)\cdots(-4-(n+1))(1-x)^{(-4-n)}(-1)$$

$$= -4!\,(1-x)^{-4-n}(-1)$$

$$= (-1)^n \; 4!(1-x)^{-4-n}$$

So

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 4!(1-x)^{-4-n}}{n!}(x)^n$$

5. Find a power series representation of the function f defined by

$f(x) = \dfrac{1}{(1-x)^4}$ centered at a = 2.

$f(x) = \left(\frac{1}{1-x}\right)^4$ or $f(x) = (1-x)^{-4}$

The common geometric series of $\frac{1}{1-x}$:

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$\left(\frac{1}{1-x}\right)^4 = \left[1 + x + x^2 + \ldots\right]^4$

$= 1 + (x)^4 + (x^2)^4 + \ldots$

$= \sum_{k=0}^{\infty} x^{4k}$

Or maybe?

$= \left[\sum_{k=0}^{\infty} x^k\right]^4$

6. Find the Taylor series for $f(x) = \ln(2+x)$ with center a = −1. Find the interval of convergence.

Check out some derivatives:

$=0$ $f(x) = \ln(2+x)$

$=1$ $f'(x) = (2+x)^{-1}$ ← Pattern emergos after $n=1$

$f''(x) = -1(2+x)^{-2}$

$f'''(x) = -2 \cdot -1 (2+x)^{-3}$

$f^{4}(x) = -3 \cdot -2 \cdot -1 (2+x)^{-4}$

$f^{n}(x) = (-1)^{n-1}(n-1)!\,(2+x)^{-n}$

Plug in our center of -1

$f^{n}(-1) = (-1)^{n-1}(n-1)!\,(2-1)^{-n}$

$= (-1)^{n-1}(n-1)!$

So our Taylor series looks like:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x+1)^{n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x+1)^{n}$$

Lets check interval of convergence with the ratio test:

$$\lim_{n\to\infty}\left|\frac{\cancel{(-1)^n}\cancel{(x+1)^{n+1}}}{(n+1)}\cdot\frac{n}{\cancel{(-1)^{n-1}}\cancel{(x+1)^n}}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(x+1)\,n}{n+1}\right|$$

$$=\lim_{n\to\infty}|x+1|\left(\frac{n}{n+1}\right)$$

$$=|x+1|$$

ASIDE

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$=\lim_{n\to\infty}\frac{1}{1+\cancel{\frac{1}{n}}^{\,0}}$$

$$=1$$

So our interval is

$$|x+1|<1$$

$$-1<x+1<1$$

$$-2<x<0$$

$\therefore$ The function $f(x)=\ln(2+x)$ can be represented by the Taylor series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x+1)^n$ on the interval $-2<x<0$.

7. Find a power series representation and determine the interval of convergence y = sech x at a = ln(2).

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

Consider the Taylor series for

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Lets adapt that to work with sech x

$$\operatorname{sech} x = \frac{2}{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] + \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right]}$$

$$= \frac{2}{2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots}$$

$$= \frac{1}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

$$= \sum_{k=0}^{\infty} \frac{1}{\frac{x^{2k}}{k!}}$$

8. Use a power series representation to calculate the following limits and integrals:

a. $\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

c. $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

b. $\int \sqrt{1 + x^3}\, dx$

d. $\int \arctan(x^2)\, dx$

8a) Known series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Lets plug this in:

$$\lim_{x \to 0} \frac{\left[\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] - \cancel{x} + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x \to 0} \left[\left[-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] + \frac{1}{6}x^3\right] \cdot \left[\frac{1}{x^5}\right]$$

$$= \lim_{x \to 0} \left[\cancel{-\frac{1}{3!}x^{-2}} + \frac{1}{5!} - \cancel{\frac{1}{7!}x^{2}} + \cdots\right] + \cancel{\frac{1}{6}x^{-2}}$$

$$\boxed{= \frac{1}{5!}}$$

8c) $\lim_{x \to 0} \frac{\tan x - x}{x^3}$

$$\tan x = \frac{\sin x}{\cos x}$$

Using the known series:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

So: $$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots}$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)}$$

Note: this fits the common $\overset{\text{Geometric}}{\downarrow}$ series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

with $x = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$

So:

$$\tan x = \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right]\left[1 + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2 + \cdots\right]$$

Now:

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(1 + \left[\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right] + \left[\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right]^2 + \cdots\right) - x}{x^3}$$

$$= \lim_{x \to 0} \left(x^{-2} - \frac{1}{3!} + \frac{x^2}{5!} - \cdots\right)\left(x^{-3} + \left[\frac{1}{2!}x^{-1} - \frac{x}{4!} + \cdots\right] + \left[\frac{1}{2!}x^{-1} - \frac{x}{4} + \cdots\right]^2 + \cdots\right) - x$$

Hmm...

8d) $\int \tan^{-1}(x^2)\,dx$

Note:

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Weirdly almost the same as $\sin x$ sans the factorial in the denomenator!

so:

$$\tan^{-1}(x^2) = (x^2)^1 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \cdots$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x^2)^{2k-1}}{k}$$

Integrate

$$\int \tan^{-1}(x^2)\,dx = \int \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x^2)^{2k-1}}{k}$$

$$\boxed{= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(x^2)^{2k}}{k+1}}$$

9. Approximate *f* by a Taylor polynomial with degree *n* at the number *a*. Use Taylor's Inequality to estimate the accuracy of the approximation of *f* when *x* lies in the given interval.

   a. $f(x) = \ln(1 + 2x)$, $a = 1$, $n = 3$, $0.5 \leqslant x \leqslant 1.5$

   b. $f(x) = e^{x^2}$, $a = 0$, $n = 3$, $0 \leqslant x \leqslant 0.1$

9a) $f(x) = \ln(1+2x)$, $a = 1$, $n = 3$,

$$\frac{1}{2} \leq x \leq \frac{3}{2}$$

Lets look at some derivatives:

$$f'(x) = \frac{1}{1+2x}(2) \qquad f'(1) = \frac{2}{3}$$

$$f''(x) = -\frac{2}{(1+2x)^2}(2) \qquad f''(1) = \frac{-4}{9}$$

$$f'''(x) = \frac{8}{(1+2x)^3}(2) \qquad f'''(1) = \frac{16}{27}$$

$$f^4(x) = -\frac{48}{(1+2x)^4}(2) \qquad f^4(1) = \frac{-96}{81}$$

So lets build our 3rd degree Taylor polynomial

$$P_3(x) = f(1) + f'(1)(x-c) + \frac{f''(1)(x-c)^2}{2!} + \frac{f'''(1)(x-c)^3}{3!}$$

Lets plug in what we know, and use our center of $c = 1$

$$P_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4}{9}\frac{(x-1)^2}{2!} + \frac{16}{27}\frac{(x-1)^3}{3!}$$

$$\boxed{= \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{27}\frac{(x-1)^3}{3}}$$

Lets find the error:

$$R_3(x) = \frac{f^{3+1}(z)(x-c)^{3+1}}{(3+1)!}$$

Remember:

$$c \le z \le x$$

$z = \frac{3}{2}$ or $z = \frac{1}{2}$

$$f^4\left(\frac{3}{2}\right) = \frac{-96}{(1+3)^4}$$

$$= \frac{-96}{256}$$

$$= -\frac{3}{8}$$

$$f^4\left(\frac{1}{2}\right) = \frac{-96}{(1+1)^4}$$

$$= \frac{-96}{16}$$

$$= -6$$

Lets use $z = \frac{1}{2}$

$$R_3(x) = \frac{-6(x-c)^4}{4!}, \quad c = 1, \; x = \frac{1}{2}$$

$$= \frac{-6\left(\frac{1}{2}-1\right)^4}{4!}$$

$$= \frac{-6\left(\frac{1}{16}\right)}{4!} = -\frac{\cancel{6}}{8} \cdot \frac{1}{4 \cdot \cancel{3} \cdot 2 \cdot 1}$$

$$\boxed{= -\frac{1}{64}}$$

$\therefore$ The function $f(x)=\ln(1+2x)$ can be estimated by the Taylor polynomial:

$$P_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{27}\frac{(x-1)^3}{3!}$$

with an error of:

$$R_3(x) = -\frac{1}{64}$$

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b) $f(x)=e^{x^2}$, $a=0$, $n=3$, $0 \leq x \leq \frac{1}{10}$

Using the known series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Lets adapt that to our function:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2n}}{n!}$$

Lets prepare our Taylor polynomial at the 3rd degree:

$$P_3(x) = f(0) + f'(0)(x-c) + \frac{f''(0)(x-c)^2}{2!} + \frac{f'''(0)(x-c)^3}{3!}$$

Lets do some derivatives :

$$f'(x) = e^{x^2}(2x)$$

$$f''(x) = e^{x^2}(2x)(2x) + e^{x^2}(2)$$

$$= e^{x^2}(4x^2 + 2)$$

$$= e^{x^2}4x^2 + e^{x^2}2$$

$$f'''(x) = e^{x^2}(2x)4x^2 + e^{x^2}(8x) + e^{x^2}(2x)2 + e^{x^2}(0)$$

$$= e^{x^2}(8x^3 + 8x + 2x)$$

$$= e^{x^2}(8x^3 + 10x)$$

So our polynomial will look like:

$$P_3(x) = e^{x^2} + e^{x^2}(2x)(x-c) + \frac{e^{x^2}(4x^2+2)(x-c)^2}{2!} + \frac{e^{x^2}(8x^3+10x)(x-c)^3}{3!}$$

Note:

$$f(0) = e^0 = 1$$

$$f'(0) = e^0(0) = 0$$

$$f''(0) = e^0(0+2) = 2$$

$$f'''(0) = e^0(0+0) = 0$$

so:

$$P_3(0) = 1 + 0 + \frac{\cancel{2}(x-c)^2}{\cancel{2!}} + 0$$

$$= 1 + (x-c)^2$$

Lets check error R_k

$$f^4(x) = e^{x^2}(2x)(8x^3+10x) + e^{x^2}(24x^2+10)$$
$$= e^{x^2}[16x^4 + 20x^2 + 24x^2 + 10]$$
$$= e^{x^2}[16x^4 + 44x^2 + 10]$$

$$f^4(0) = e^0[10] = 10$$

$$f^4(\tfrac{1}{10}) = e^{(\frac{1}{10})^2}[16(\tfrac{1}{10})^4 + 44(\tfrac{1}{10})^2 + 10] = e^{1/100}\left[\frac{5003}{500}\right]$$

ASIDE

$$\frac{10}{4!} = \frac{\cancel{10}\,5}{4\cdot3\cdot2\cdot1} = \frac{5}{12}$$

So:

$$R_k(x) = \frac{10(x-c)^4}{4!} = \frac{5(x-c)^4}{12}$$

let $x = \frac{1}{20}$

let $z = \frac{1}{10}$

$$R_k(\tfrac{1}{20}) = \frac{e^{1/100}\left[\frac{5003}{500}\right](\frac{1}{20})^4}{4!}$$

$\approx .0000026$

Hmm, seems too accurate...