$k^2 - k\sqrt{3}$

Zed Chance

1. Use the integral test to prove $\sum_{k=1}^{\infty} \frac{1}{k(k-\sqrt{3})}$ converges.

$$f(x) = \frac{1}{x(x-\sqrt{3})}$$

On interval $[1, \infty)$ $f(x)$ is positive, and continuous.

$$f'(x) = \frac{-1(2x - \sqrt{3})}{(x^2 - x\sqrt{3})^2}$$

$f'(x)$ is decreasing on our interval. (always negative)

so:

$$\sum_{k=1}^{\infty} \frac{1}{k(k-\sqrt{3})} = \int_1^{\infty} \frac{1}{x(x-\sqrt{3})}\, dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{1}{x(x-\sqrt{3})}\, dx$$

$$= \lim_{t\to\infty} \int_1^t \frac{A}{x} + \frac{B}{x-\sqrt{3}}\, d$$

$$1 = A(x-\sqrt{3}) + B(x)$$

let $x = \sqrt{3}$

$1 = \sqrt{3}\, B$

$B = \frac{\sqrt{3}}{3}$

let $x = 0$

$1 = -\sqrt{3}A$

$A = -\frac{\sqrt{3}}{3}$

$$= \lim_{t \to \infty} \int_1^t \frac{-\frac{\sqrt{3}}{3}}{x} + \frac{\frac{\sqrt{3}}{3}}{x - \sqrt{3}} \, dx$$

$$= \lim_{t \to \infty} \left[-\frac{\sqrt{3}}{3} \ln|x| + \frac{\sqrt{3}}{3} \ln|x - \sqrt{3}| \right]_1^t$$

$$= \lim_{t \to \infty} \frac{\sqrt{3}}{3} \left[-\ln|t| + \ln|t - \sqrt{3}| - \left(\cancelto{0}{\ln|1|} + \ln|1 - \sqrt{3}| \right) \right]$$

$$= \lim_{t \to \infty} \frac{\sqrt{3}}{3} \left[-\ln|t| + \ln|t - \sqrt{3}| - \ln|1 - \sqrt{3}| \right]$$

$$= \lim_{t \to \infty} \sqrt{3} \left[\cancelto{0}{-\frac{\ln|t|}{3}} + \cancelto{0}{\frac{\ln|t - \sqrt{3}|}{3}} - \frac{\ln|1 - \sqrt{3}|}{3} \right]$$

$$\boxed{= \frac{-\sqrt{3} \ln|1 - \sqrt{3}|}{3}}$$

Since the integral converges, the series must also converge!

2. Find the value of $\sum_{k=10}^{\infty} 3^{k+2}4^{3-k}$

$$S_n = \left(3^{12}4^{-7}\right) + \left(3^{13}4^{-6}\right) + \left(3^{14}4^{-5}\right) + \dots$$

$$\left(3^{(k-1)+2}4^{3-(k+1)}\right) + \left(3^{k+2}4^{3-k}\right)$$

$$\sum_{k=10}^{\infty} 3^{k+2}4^{3-k}$$

$$= \sum_{k=10}^{\infty} 3^{12}3^{k-10}4^{3-k}$$

$$= \sum_{k=10}^{\infty} 3^{12}4^{3}3^{k-10}4^{-k}$$

$$= \sum_{k=10}^{\infty} \frac{3^{12}4^{3}}{4^{k}} 3^{k-10}$$

$$= 3^{12}4^{3} \sum_{k=10}^{\infty} \frac{3^{k-10}}{4^{k}}$$

$$= 3^{12}4^{3} \sum_{k=10}^{\infty} \left(\frac{3^{1-\frac{10}{k}}}{4}\right)^{k}$$

Geometric?

3. Use your knowledge of geometric series to write $10.1\overline{35}$ as a ration of two integers.

$$= 10 + \frac{1}{10} + \frac{7}{200} + \frac{7}{20000} + \cdots$$

$$= \frac{101}{10} + \sum_{k=0}^{\infty} \left(\frac{7}{20}\right)\left(\frac{1}{10}\right)^{2k+1}$$

$$= \frac{101}{10} + \frac{\frac{7}{20}}{1 - \frac{1}{10}}$$

$$= \frac{101}{10} + \frac{7}{20} \frac{10}{9}$$

$$= \frac{101}{10} + \frac{7}{18}$$

$$\boxed{= \frac{472}{45}}$$

4. Why would you not want to use IT on $\sum_{k=1}^{\infty} e^{-k^2}$?

✓ Continuous
✓ Positive
✓ Decreasing

$S_n = e^{-1} + e^{-4} + e^{-9} \ldots + e^{-(k-1)^2} + e^{-k^2}$

Maybe because $\int e^{-x^2}\,dx$ is hard to do?

5. $\sum_{3}^{\infty}\left(\frac{1}{\ln k}-\frac{1}{\ln(k+2)}\right)$ is a telescoping series. Determine if the series converges.

$$S_k = \left(\frac{1}{\ln 3} - \cancel{\frac{1}{\ln(5)}}\right) + \left(\cancel{\frac{1}{\ln 4}} - \frac{1}{\ln 6}\right) + \left(\cancel{\frac{1}{\ln 5}} - \frac{1}{\ln 7}\right) + \ldots$$

$$\cancel{\left(\frac{1}{\ln k-2} - \frac{1}{\ln k}\right)} + \left(\frac{1}{\ln(k-1)} - \cancel{\frac{1}{\ln(k+1)}}\right) + \left(\cancel{\frac{1}{\ln k}} - \frac{1}{\ln(k+2)}\right)$$

$$S_k = \frac{1}{\ln 3} - \frac{1}{\ln(k+2)}$$

$$= \lim_{k\to\infty}\left[\frac{1}{\ln 3} - \frac{1}{\ln(k+2)}\right]$$

$$\boxed{= \frac{1}{\ln 3}}$$

6. Determine the value of $\lim_{n\to\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+2)}\right)$. Hint: You might be able to use #4 if the series converges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\ln n}-\frac{1}{\ln(n+2)}\right)$$

$$S_n=\left(\frac{1}{\ln 1}-\cancel{\frac{1}{\ln 3}}\right)+\left(\cancel{\frac{1}{\ln 2}}-\cancel{\frac{1}{\ln 4}}\right)+\left(\cancel{\frac{1}{\ln 3}}-\cancel{\frac{1}{\ln 5}}\right)+\dots$$

$$\left(\cancel{\frac{1}{\ln(n-2)}}-\cancel{\frac{1}{\ln n}}\right)+\left(\cancel{\frac{1}{\ln(n-1)}}-\cancel{\frac{1}{\ln(n+1)}}\right)+\left(\cancel{\frac{1}{\ln n}}-\frac{1}{\ln(n+2)}\right)$$

$$S_n=\left(\cancel{\frac{1}{\ln 1}}^{0}-\frac{1}{\ln(n+2)}\right)$$

$\Downarrow$

not continuous on interval...

lets try integral test on $[2,\infty)$:

$$f(x)=\frac{1}{\ln x}-\frac{1}{\ln(x+2)}$$

$$\int_2^{\infty}\frac{1}{\ln x}-\frac{1}{\ln(x+2)}\,dx$$

$$=\lim_{b\to\infty}\int_1^{b}\frac{1}{\ln x}-\frac{1}{\ln(x+2)}\,dx$$

$\searrow$

$$= \lim_{b \to \infty} \int_1^b \frac{1}{\ln x} - \frac{1}{\ln(x+2)}\, dx$$

$$= \lim_{b \to \infty} \left[\underbrace{\int_1^b \frac{1}{\ln x}\, dx}_{I_1} - \underbrace{\int_1^b \frac{1}{\ln(x+2)}\, dx}_{I_2} \right]$$

$$I_1 = \int \frac{1}{\ln x}\, dx \qquad \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\, dx \end{array} \qquad \begin{array}{l} dv = dx \\ v = x \end{array}$$

$$= x \ln x - \int x \frac{1}{x}\, dx$$

$$= x \ln x - x$$

$$I_2 = \int \frac{1}{\ln(x+2)}\, dx \qquad \begin{array}{l} u = x+2 \\ du = dx \end{array}$$

$$= \int \frac{1}{\ln u}\, dx$$

$$= u \ln u - u$$

$$= (x+2) \ln(x+2) - x+2$$

$$= x+2 \left(\ln(x+2) - 1 \right)$$

SO:

$$\lim_{b \to \infty} \Big[x \ln x - x - \big(x+2 (\ln(x+2) - 1) \big) \Big]_1^b$$

$$= \lim_{b \to \infty} \Big[x(\ln x - 1) - (x+2)(\ln(x+2) - 1) \Big]_1^b$$

$$= \lim_{b \to \infty} \Big[b(\ln b - 1) - (b+2)(\ln(b+2) - 1) \Big] - \Big[(\overset{0}{\ln 1} - 1) - 3(\ln(3) - 1) \Big]$$

$$\boxed{= \infty \qquad \text{DIVERGES!}}$$

7. A student wants to use IT on the series $\sum_{k=1}^{\infty}\left[e^{-\ln k}+\cot\left(\frac{\pi}{2}(2k+1)\right)\right]$ by defining the function f such that $f(x)=\frac{1}{x}$. Can she do that? Could she also use $f(x)=e^{-\ln x}+\cot\left(\frac{\pi}{2}(2x+1)\right)$.

If $\frac{1}{x}$ models the sequence then yes she can.

She can also use the second equation.

8. Prove $\sum_{k=1}^{\infty} \frac{1}{k^3}$ using the integral test, series comparison test, the limit comparison test, and the ratio test. Which was the easiest to establish and implement?

IT: $f(x) = \frac{1}{x^3}$

$$\int_1^{\infty} \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2}\,dx$$

$$= \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\cancel{\frac{1}{2b^2}}^{\,0} - \left(-\frac{1}{2}\right)\right]$$

$$\boxed{= -\frac{1}{2} \quad \text{convergence}}$$

Comparison:

$$0 \le \frac{1}{x^3} \le \frac{1}{x^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2$

Since $p > 1$ it means that the series converges. Since the upper series converges the lower must also.

Limit comparison:

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \quad \text{compared to} \quad \frac{1}{k^2}$$

$$\frac{1}{k^{21}} \; k^{2}$$

$$\lim_{k\to\infty} \frac{\frac{1}{k^3}}{\frac{1}{k^2}}$$

$$= \lim_{k\to\infty} \frac{1}{k}$$

$$= 0$$

$n^2 = n^1 \cdot n^1$

$b^n = b \cdot b^{n-1}$

$b^{n^2} = b^n \cdot b^n$

$= b^n \cdot b \cdot b^{n-1}$

$= b^n \cdot b^{n-1}$

9. Determine if the following series converges:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n} \left(\frac{n}{n+1}\right)^{n-1}$$

Geometric with $a = \left(\frac{n}{n+1}\right)^n$

$r = \frac{n}{n+1}$

r is always < 1

$$\boxed{\therefore = \frac{\left(\frac{n}{n+1}\right)^n}{1 - \frac{n}{n+1}}}$$

10. Determine if the following series converges:

$$\sum_{k=1}^{\infty} \frac{1}{4+e^{-k}}$$

$e^{-k} = \frac{1}{e^k}$

$$\lim_{k \to \infty} \frac{1}{4+e^{-k}} = \frac{1}{4}$$

Since lim of som isn't 0, the series diverges!

Zed Chance

11. Determine if the following series converges:

$$\sum_{k=1}^{\infty} e^{-k^2}$$

$$\sum_{k=1}^{\infty} \frac{1}{e^{k^2}}$$

$$\lim_{k\to\infty} \left| \frac{\frac{1}{e^{(k+1)^2}}}{\frac{1}{e^{k^2}}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{e^{k^2}}{e^{(k+1)^2}} \right|$$

$$k^2 - (k^2 + 2k + 1)$$

$$2k + 1$$

$$L = \lim_{k\to\infty} \left| e^{2k+1} \right|$$

$$\boxed{L > 1 \therefore \text{the series diverges}}$$