

MATH170

Linear Programming

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1 A Brief Introduction

Consider n input or decision variables for an optimization (maximization or minimization) problem, say x_1, x_2, \dots, x_n . Let $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ be a linear combination of these input variables, where c_1, c_2, \dots, c_n are some scalars (real numbers). Aug 31

A linear programming problem (LPP) is an optimization problem of such a linear expression under linear constraints for the input variables. For demonstration purposes and geometric convenience, let's first consider the two-dimensional case that involves two decision variables x and y (for optimizing a function of two variables).

Definition 1 (Feasible region). Let $f(x, y) = ax + by$ be a function of two variables to be optimized. It is usually called the **object function** of LPP.

Some examples of linear constraints would be

- $cx + dy = e$, equality restraint
- $cx + dy \leq e$ or $cx + dy \geq e$, inequality restraint

There would be one or more of such linear constraints in any LPP. The region that is determined by the intersection of all these constraints is called the **feasible region**.

For now, we are not going to consider strict inequalities ($<$ or $>$) in the constraints to ensure existence of optimal solutions.

If no point satisfies all the constraints, then the problem is not feasible.

1.1 Method of Corners

Let's now study a numerical example in two decision variables by first describing its object function, constraints and feasible region. Then, we solve it using a geometric approach (method of corners).

Example 1. The loan department of a bank has a total limit of 60 million dollars to allocate for home(owner) and car loans. Assume that annual rate of return is roughly 6% for home loans, and 8% for auto loans, on average.

Bank management imposes a restriction that the amount of home loans should be at least three times that of the car loans.

Determine the optimal amounts of home and car loans to maximize the overall annual return from the total allocation.

Let x = the amount of home loans, and y = amount of auto loans. Let z = the return amount. So, our objective function is

$$z = f(x, y) = 0.06x + 0.08y$$

We have a constraint of 60 million dollars allocated to the loan department, and we can't have a negative amount of loans.

$$x \geq 0 \qquad y \geq 0 \qquad x + y \leq 60$$

The amount of home loans has to be 3 times the amount of car loans, so

$$x \geq 3y \Rightarrow 3y - x \leq 0$$

If we graph all of these lines, the feasible region is the intersection of all constraints. The feasible region R is a triangle with vertices $(0, 0)$, $(60, 0)$, $A = (?, ?)$. A is the intersection of the lines $3y - x = 0$ and $x + y = 60$. So, $A = (45, 15)$

We can check the value of z at the corners of the triangle R to determine the optimal value.

$$\begin{aligned} f(0, 0) &= 0 \\ f(60, 0) &= 0.06(60) + 0 = 3.6 \\ f(45, 15) &= 0.06(45) + 0.08(15) = 3.9 \end{aligned}$$

So, the optimal amount of loans is \$45 million in home loans and \$15 million in auto loans.

The **method of corners** relies on the fact that an optimal value is guaranteed to occur at a corner point in the feasible region. Sep 02

1.2 Method of Parallel Lines

One way to graphically justify that an optimal value of a feasible LPP in two variables occurs at a corner point is to use the **method of parallel lines**. The parallel line is also called the “isoprofit line”.

Note that the value of an object function $z = ax + by$ is constant on a line segment $ax + by = z_0$ in the feasible region where z_0 is constant.

For each (x, y) in R and z_0 , which is accessible, the value of z is constant on the line segment $ax + by = z_0$ ($z = z_0$). This can also be denoted as

$$y = \frac{z_0}{b} - \frac{a}{b}x$$

where the slope is fixed (and is usually negative in a maximization problem).

We can consider lines parallel to the line described above (we’ll call it ℓ_1).

So for a maximization problem, we can change the y -intercept but not the slope (find parallel lines to ℓ_1) to a larger value.

Example 2. See notes for graph

Lets consider maximizing the function

$$z = ax + by$$

in the feasible region R .

A parallel line ℓ to ℓ_1 will have a larger y -intercept if we find a parallel line in the north eastern direction. We observe that ℓ_2 is the corresponding optimal line that touches R at one point (a corner point).

So, the y -intercept of ℓ_2 is the maximum possible within the feasible region R .

Note 1. Similarly, we can move line ℓ_1 in the south western direction for a minimization problem (with a negative slope).

Example 3. See notes for graph

Lets solve the home/auto loan problem using the method of parallel lines. Recall,

$$z = 0.06x + 0.08y$$

For (x, y) in R at possible z_0 values, our isoprofit line is

$$\begin{aligned} 0.06x + 0.08y &= z_0 \\ y &= \frac{z_0}{0.08} - \frac{0.06}{0.08}x \\ &= \frac{z_0}{0.08} - \frac{3}{4}x \end{aligned}$$

For example, consider an interior point, $C = (45, 10)$:

$$z = 0.06(45) + 0.08(10) = 3.5$$

which is not optimal since we can clearly increase either x (to 50) or y (to 15).

Lets use parallel lines starting with the line that passes through $C = (45, 10)$:

$$\ell_0 = y = \frac{3.5}{0.08} - \frac{3}{4}x$$

where the slope is $-\frac{3}{4}$.

We can draw a parallel line ℓ_1 which passes through point $A = (45, 15)$. The corresponding isoprofit line is

$$\ell_1 = y = \frac{z_1}{0.08} - \frac{3}{4}x$$

where $z_1 = 3.9$.

So, the optimal solution is $x = 45$ and $y = 15$.

Remarks

- If there is a solution to a LPP with two variables, then one of the solutions occurs at a corner point.
- An optimal solution is not guaranteed for a LPP, especially when the feasible region is unbounded.
- It is possible to have multiple solutions, (x, y) values, which occurs when the optimizing line is parallel with one of the boundary lines of the feasible region.

Example 4. Consider the previous home/auto loan problem with the same constraints, but lets change the rates of the loans to create a new objective function:

$$z = 0.05x + 0.05y$$

Show that the LPP of maximizing annual return has infinitely many solutions.

Since our function has a slope of -1 , and our constraint $x + y \leq 60$ also has a slope of -1 , we have a isoprofit line parallel to one side of our feasible region. Thus, any point on that side of the feasible region R will give our optimal solution, and so there are infinitely many solutions.

Example 5. See *LP Ex1.pdf*

Minimize $z = 3x + y$ subject to the following system of linear inequalities:

$$5x + 2y \geq 40$$

$$x + 2y \geq 20$$

$$x \geq 0$$

$$y \geq 3$$

While R is an unbounded region, seen in Figure 1.1, since we are doing a minimization problem it should still have a minimum solution. Lets check the corners of R

$$f(0, 20) = 3(0) + 20 = 20$$

$$f(5, 7.5) = 3(5) + 7.5 = 22.5$$

$$f(14, 3) = 3(14) + 3 = 45$$

So, the corner $(0, 20)$ is the optimal value to minimize this function.

Lets use the method of parallel lines. Given (x, y) in R and a possible z_0 value,

$$z_0 = 3x + y$$

has as slope of -3 since

$$\ell_0 = y = z_0 - 3x$$

We can shift the line south west, until it only touches a single point of R , the point $(0, 20)$. So,

$$z = 3(0) + 20 = 20$$

is the minimum value.

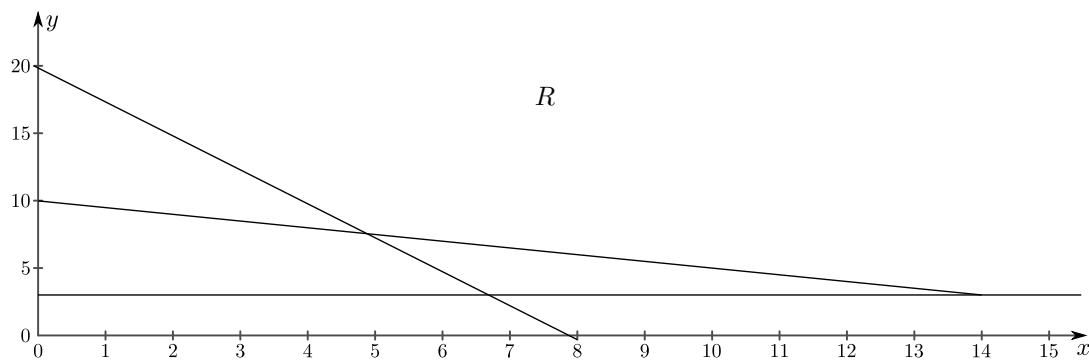


Figure 1.1: Unbounded feasible region

Example 6. A company manufactures two products, A and B , on two machines, I and II . It has been determined that the company will realize a profit of \$4/unit of product A and \$5/unit of product B .

To manufacture a unit of product A requires 6 minutes on machine I and 6 minutes on machine II . To manufacture a unit of product B requires 8 minutes on machine I and 5 minutes on machine II .

There are 4 hours of machine time available on machine I and 4 hours of machine time available on machine II in each shift. The company's goal is to maximize profit.

1. Describe this problem as LP problem by indicating the objective function and the constraints explicitly. Simplify the equations as much as possible.

We can describe this using a table:

	Product A	Product B	Time available
Unit profit	4	5	
Time on machine I	6	8	240 min
Time on machine II	6	5	180 min

So, to maximize total profit

$$z = 4x + 5y$$

subject to some constraints:

$$\begin{aligned} 6x + 8y &\leq 240 & 6x + 5y &\leq 180 \\ x &\geq 0 & y &\geq 0 \end{aligned}$$

So our objective function is

$$z = f(x, y) = 4x + 5y$$

2. Solve the LP problem you identified in the parts above using the method of corners. Indicate the optimal profit and optimal production levels in each shift.

See Figure 1.2 for graph

Solve for the intercepts in the constraint lines from previous step.

We can check direction of the inequalities to find the feasible region. The corners defined are the y -intercept, x -intercept, and the intersection between the two lines. To find that point find the intersection of the constraint lines, so our points to check are

$$\begin{aligned} D &= (0, 30) \\ B &= \left(13\frac{1}{3}, 20\right) \\ C &= (30, 0) \end{aligned}$$

and the origin $(0, 0)$. So,

$$\begin{aligned} f(0, 0) &= 0 \\ f\left(\frac{40}{3}, 20\right) &= 4\left(\frac{40}{3}\right) + 5(20) = 153\frac{1}{3} \\ f(30, 0) &= 4(30) + 5(0) = 120 \\ f(0, 30) &= 4(0) + 5(30) = 150 \end{aligned}$$

So the optimal value is found via point B , so the profit per shifts are maximized when $\frac{40}{3}$ units of product A and 20 units of product B are manufactured. Then the maximum profit would be $z = \$153\frac{1}{3}$ per shift.

3. What if we are restricted to producing whole units (integer values) for both x and y ? We will revisit this integer programming problem later.

The new optimal value is usually close to the optimal solution (without integer constraints). In our case, our above optimal solution was $(x, y) = (13\frac{1}{3}, 20)$. So we can try integer values around this solution, for example the points $(13, 20)$, $(14, 19)$, etc. We must check that these points are feasible, and satisfy the constraints. Now, we can check the z values for these points:

$$f(13, 20) = 4(13) + 5(20) = 152$$

$$f(14, 19) = 4(14) + 5(19) = 151$$

So, under the integer constraints, the new optimal solution is $(x, y) = (13, 20)$ with the maximal profit of $z = \$152$ per shift.

Note 2. This process is more involved in multiple variables (higher dimensions), and we will need to use technology to solve them.

Remark We can adjust the objective function and the constraints if an additional variable is added to a LPP. For example, if a third product, say product C , is to be manufactured with a unit product of \$6/unit, then the new objective function would be

$$z = 4x_1 + 5x_2 + 6x_3$$

Then, we also revise the constraints based on machine times for each product.

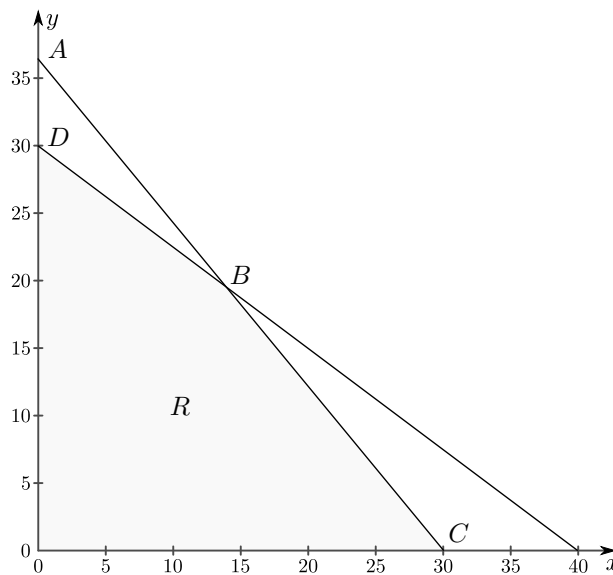


Figure 1.2: Manufacturing feasible region

2 Simplex Algorithm

We can use the simplex algorithm to solve LPP. We utilize some procedures which are similar to the methods of linear algebra, including: row reductions, Gauss-Jordan elimination, pivoting, etc. We will start with converting constraints with inequalities to constraints with equalities by introducing some additional variables (slack variables, excess/surplus variables, free variables, etc).

2.1 Standard form

In this section, we will only consider non-negative variables (we'll apply the "sign" constraints).

- For " \leq " inequality type constraints, we will add a slack variable to the left side to obtain "=" equality constraints.
- For " \geq " inequality type constraints, we will subtract excess (or surplus) variables from the left side to get "=" equality constraints.
- We will consider non-negative scalars/constants on the right side of these constraint expressions.

These assumptions/procedures give us a form of the LPP which is called the **standard** or canonical form.

Example 7. Lets apply these to the previous example. Recall, maximize $z = 4x + 5y$ subject to the constraints

$$6x + 5y \leq 180$$

$$6x + 8y \leq 240$$

and x and y are non-negative.

To get into standard form, we will need to add slack variables to the left side to make sure they become an equality. (In the case that the slack variable will make the value of the left side become bigger than the right, we can think of the slack variable being 0 so the constraint still holds).

$$6x + 5y + s_1 = 180$$

$$6x + 8y + s_2 = 240$$

where s_1, s_2 are non-negative.

The new objective function is

$$z' = 4x + 5y + 0s_1 + 0s_2$$

So to maximize z' , subject to the constraints, we can describe everything using matrix/vector notation:

$$\bar{u} = \begin{bmatrix} x \\ y \\ s_1 \\ s_2 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 180 \\ 240 \end{bmatrix} \quad A = \begin{bmatrix} 6 & 5 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{bmatrix}$$

So the full objective z' can be written as

$$z' = \bar{c} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 0 \end{bmatrix} \circ \bar{u}^T$$

subject to $A\bar{u} = \bar{b}$ and $\bar{u} \geq \bar{0}$.

Example 8. Maximize the function $z = 20 - x - y$, subject to the constraints

$$\begin{aligned} -3x + 3y &\leq 6 \\ -8x + 4y &\leq 4 \end{aligned}$$

where x and y are non-negative.

Right away you can tell that the origin $(0, 0)$ is in the feasible region by attempting to plug them in. If we solve the 3 equations for y , and graph the lines, we can see that the feasible region R is unbounded. We can then find the optimal value using a geometric approach, like before.

But lets use slack variables $s_1 \geq 0$, and $s_2 \geq 0$. We can rewrite the constraints as:

$$\begin{aligned} -3x + 3y + s_1 &= 6 \\ -8x + 4y + s_2 &= 4 \end{aligned}$$

Next, we can rewrite the objective function z with the new variables:

$$z' = 20 - x - y + 0s_1 + 0s_2$$

So now we can try to maximize z' subject to the new constraints (remember, all the variables are non-negative). This is now a LPP in standard form (canonical form) because it meets the following criteria:

1. All decision variables are non-negative.
2. Except the previous sign constraints, all constraints should be stated as equalities.
3. Right hand side coefficients of constraints are non-negative.
4. Each constraint has an isolated variable with a +1 coefficient. Moreover, such an isolated variable should not appear in another constraint. These variables also appear in the objective function with a zero coefficient.

In our case, s_1 and s_2 are the isolated variables. Such variables that satisfy all these conditions are called **basic variables**.

As a first attempt to solve a LPP in standard form, we set the remaining variables (non basic variables) equal to 0, and obtain an initial solution. This initial solution is called a **basic solution**. If it is also feasible, then we call it a **basic feasible solution** (BFS).

Idea Check if the first BFS is optimal. If not, then use some linear operations (row reduction, substitution, elimination, pivoting, etc.) to obtain another BFS. On each iteration, this procedure provides a new BFS which corresponds to a corner point in the feasible region. We can continue this until an optimal value is reached, or there is no solution.

So, now to actually get our solution now that it is in standard form. The first BFS can be found by setting the non-basic variables to 0:

$$\begin{aligned} 0 + 0 + s_1 &= 6 \\ s_1 &= 6 \\ 0 + 0 + s_2 &= 4 \\ s_2 &= 4 \end{aligned}$$

So, $(x, y, s_1, s_2) = (0, 0, 6, 4)$ is our BFS. We can find the corresponding z' :

$$\begin{aligned} z' &= 20 - x - y + 0s_1 + 0s_2 \\ &= 20 - 0 - 0 - 0 - 0 \\ &= 20 \end{aligned}$$

So is this optimal? Since $z = 20 - (x + y)$ is smaller than 20 when $x > 0$ or $y > 0$ in the feasible region, see Figure 2.1, it will only subtract from the 20. So $z = 20$ is actually the optimal value and $(x, y) = (0, 0)$ is the optimal solution to the original problem.

This problem satisfies some conditions that ensures an optimal solution to a maximization problem.

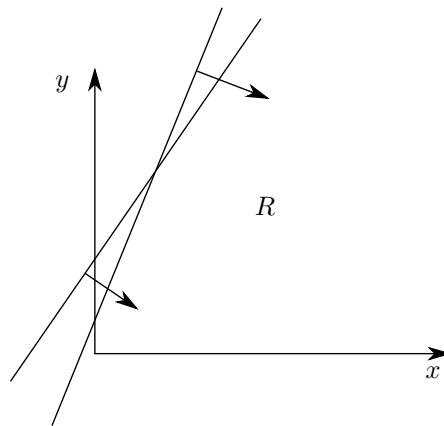


Figure 2.1: A look at the feasible region

Definition 2 (Optimality criterion for a max LPP). Assume that in a maximization LPP, every non basic variable has a non-positive coefficient in the objective function (in a canonical form). Then, a BFS given by the canonical form maximizes the objective function (it is an optimal solution).

Example 9. Lets change the objective function of the last problem to

$$z = 3x - y + 30$$

subject to the same constraints

$$\begin{aligned} -3x + 3y &\leq 6 \\ -8x + 4y &\leq 4 \end{aligned}$$

So, we also get the revised constraints with our slack variables

$$\begin{aligned} -3x + 3y + s_1 &= 6 \\ -8x + 4y + s_2 &= 4 \end{aligned}$$

So our z' is

$$z' = 3x - y + 20 + 0s_1 + 0s_2$$

Lets find our BFS by plugging in 0 for our non basic variables.

$$\begin{aligned} (x, y, s_1, s_2) &= (0, 0, 6, 4) \\ z' &= 20 \end{aligned}$$

However, since the x variable is giving a positive contribution to the z' value (compared to the last problem), we should be able to improve the output. Unfortunately though our x is unbounded in the region R , and the larger the x value the larger the z value, and $x \rightarrow \infty$, so the optimal solution doesn't exist.

Now, we can change the feasible area to find an improvement. Consider $x = 0$, and allow y to be positive. However, this implies that y should be made a basic variable in a canonical form to replace either s_1 or s_2 . So,

$$\begin{aligned} 0 + 3y + s_1 &= 6 \\ 0 + 4y + s_2 &= 4 \end{aligned}$$

Which is equivalent to

$$\begin{aligned} s_1 &= 6 - 3y \\ s_2 &= 4 - 4y \end{aligned}$$

So we have some new constraints

$$y \leq 2 \qquad y \leq 1$$

By imposing these new constraints and $y > 0$, we get

$$0 < y \leq 1$$

So lets choose $y = 1$ in this range since y has a positive coefficient in our objective function z . We can find s_1 and s_2 with $y = 1$:

$$s_1 = 3 \qquad s_2 = 0$$

So our new BFS is

$$(x, y, s_1, s_2) = (0, 1, 3, 0)$$

For this BFS, our z' value is

$$\begin{aligned} z' &= 20 - 3(0) + 1 + 0(3) + 0(0) \\ &= 21 \end{aligned}$$

so the value has improved.

Definition 3 (Unboundedness criterion). We can tell if we're in an unbounded LPP more generally by applying the following criterion:

1. Assume that a non basic variable has a positive coefficient in the objective function (in canonical form) for a max LPP.
2. The coefficients of that variable is zero or negative in each constraint (except the sign constraints).

Then, the objective function is unbounded over the feasible region: no optimal solution exists.

Remark For a given minimization problem, we can convert it to a maximization problem by changing the sign of the objective function and use the same constraints. Then, the criteria above apply the equivalent maximization problem. Sep 14

Example 10. Lets continue the last example.

We will work on a procedure to show some ratios of coefficients and the process of making y a new basic variable (using row operations and pivoting).

We can rewrite the constraints

$$\begin{aligned}c_1 &: -3x + 3y + s_1 = 6 \\c_2 &: -8x + 4y + s_1 = 4\end{aligned}$$

To determine the pivot row for y , we compute and compare the ratios of the right hand side constants to the coefficients of y (with positive values). Then, we select the row with the smallest (positive) ratio to create a +1 coefficient for y .

$$\begin{aligned}c_1 &: \text{ratio} = \frac{6}{3} = 2 \\c_2 &: \text{ratio} = \frac{4}{4} = 1 \leftarrow \text{smallest ratio}\end{aligned}$$

So the pivot row (constraint) is c_2 . Lets multiply by sides of c_2 by $\frac{1}{4}$ to get our new constraint (which is equivalent):

$$\begin{aligned}c'_2 &= \frac{1}{4}c_2 \\&= -2x + 1y + \frac{1}{4}s_2 \\&= 1\end{aligned}$$

Next, eliminate y from c_1 using c'_2

$$\begin{aligned}c'_1 &= c_1 - 3c'_2 \\&= -3x + 3y + s_1 - 3(-2x + y + \frac{s_2}{4}) \\&= 3x + s_1 - \frac{3}{4}s_2\end{aligned}$$

And since $c_1 = 6$ and $c'_2 = 1$, we have $3x + s_1 - \frac{3}{4}s_2 = 3$.

Finally, we need to write an objective function in canonical form based on the new basic variables. For that, we can substitute y in terms of other variables. From c'_2 :

$$y = 1 + 2x - \frac{1}{4}s_2$$

which we can then plug back into z'

$$\begin{aligned} z' &= 20 - 3x + y + 0s_1 + 0s_2 \\ &= 20 - 3x + \left(1 + 2x - \frac{1}{4}s_2\right) + 0s_1 + 0s_2 \\ &= 21 - x - \frac{1}{4}s_2 + 0s_1 + 0y \end{aligned}$$

which we can call z'' . We can maximize $z'' = 21 - x - \frac{1}{4}s_2 + 0s_1 + 0y$ subject to the same constraints, where the non basic variables in this new canonical form have non-positive coefficients. Therefore, the optimality principal applies.

Definition 4 (Improvement criterion). Assume that a non basic variable has a positive coefficient in the objective function. If this variable has a positive coefficient in a constraint equation in a canonical form, then a new BFS can be obtained by converting this variable into a basic variable by pivoting.

Definition 5 (Ratio and pivoting criterion). Assume that we want to convert a non basic variable x_n , to a basic variable for an improved BFS.

- We can compute the ratios of the right hand side constant to the coefficient of x_n in each constraint equation where x_n has a positive coefficient.
- We select the constraint with the smallest ratio as the pivot constraint and row.
- Then, we make the coefficient of y in that constraint equal to 1, by scaling.
- Finally, we eliminate y from other constraints.

The new system is equivalent to the original.

Each time we obtain a new candidate for a new BFS, we also revise the objective function in canonical form and apply optimality criterion to check if it is optimal. If not, we can keep applying the criteria above to obtain another solution until an optimal solution is reached. This procedure is called the **simplex method**.

2.2 Criteria

Consider a maximization LPP problem in canonical form, recall all criterion:

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- Optimality Criterion

If every nonbasic variable has a nonpositive coefficient in the objective function, then a BFS in the canonical form maximizes the objective function (so it is a optimal solution to LPP).

- Unboundedness Criterion

Assume that a nonbasic variable has a positive coefficient in the objective function (in a canonical form). If this variable has negative or zero coefficients in all constraints (except sign constraints), then the objective function is unbounded from above over the feasible region. In other words, no optimal solution to LPP exists.

- Improvement Criterion

Assume that a nonbasic variable has a positive coefficient in the objective function (in a canonical form). If this variable has a positive coefficient in a constraint equation, then a new BFS can be obtained by converting this variable into a basic variable via pivoting. The objective function's value improves with this BFS.

- Ratio and Pivoting Criterion

Assume that we want to convert a nonbasic variable into a basic variable to get a new BPS. We compute the ratio of right hand side constant to the coefficient of this variable in each constraint (where the corresponding coefficient is positive). We select the constraint with the smallest ratio to be the pivot constraint. So, we make the coefficient of the variable in this constraint +1 (normalization), and eliminate it from the other constraints.

Here's a new improvement criterion,

Definition 6 (Simplex algorithm improvement criterion). Start with a variable with the largest positive coefficient in the objective function.

- Note that after the normalization and elimination steps, we need to identify the new basic variable and revise the objective function to describe it in a canonical form (by eliminating the basic variables in the objective function via a zero coefficient).
- If there are any other nonbasic variables with a positive coefficient in the objective function, we can repeat the ideas in the improvement ratio/pivot criteria to obtain a new BFS with a corresponding improved value of the objective function (z value).
- However, there is one decision item that we haven't addressed yet: what do we do if there are multiple variables with a positive coefficient in the objective function (assuming that two or more of them satisfy the conditions of the improvement criteria)?

Example 11. Maximize $z = 6x_1 + 14x_2 + 13x_3$ subject to

$$\begin{aligned} \frac{1}{2}x_1 + 2x_2 + x_3 &\leq 24 \\ x_1 + 2x_2 + 4x_3 &\leq 60 \\ x_k &\geq 0, \text{ for each } k \end{aligned}$$

Due to the " \leq " in both constraints, we add slack variables $s_1 = x_4$ and $s_2 = x_5$ to obtain a canonical form:

$$\begin{aligned} c_1 : \frac{1}{2}x_1 + 2x_2 + x_3 + x_4 &= 24 \\ c_2 : x_1 + 2x_2 + 4x_3 + x_5 &= 60 \end{aligned}$$

with $x_k \geq 0$ for $k = 1, 2, \dots, 5$.

Note that the basic variables are x_4 and x_5 . Our nonbasic variables are x_1, x_2, x_3 . We can obtain our initial BFS by letting our nonbasic variables = 0 and solve for our basic variables x_4 and x_5 . So, let $x_1 = x_2 = x_3 = 0$ and we get:

$$\begin{aligned} x_4 &= 24 \\ x_5 &= 60 \end{aligned}$$

This gives us an objective function of

$$z = 6x_1 + 14x_2 + 13x_3 + 0x_4 + 0x_5$$

and constraints c_1 and c_2 can be described in a matrix form $A\bar{x} = \bar{b}$:

$$A = \begin{bmatrix} \frac{1}{2} & 2 & 1 & 1 & 0 \\ 1 & 2 & 4 & 0 & 1 \end{bmatrix}$$

$$\bar{b} = \begin{bmatrix} 24 \\ 60 \end{bmatrix}$$

$$\bar{x} = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5]^T$$

So our BFS is given by

$$[0 \quad 0 \quad 0 \quad 24 \quad 60]$$

with a corresponding $z = 0$. This is not an optimal solution because **the nonbasic variables have positive coefficients in the objective function**, and **there is room for improvement because there are positive coefficients in the constraints**.

So, let's use the simplex criterion for improvement, and pick the variable with the biggest positive coefficient in the objective function, which is x_2 . Then, we apply the ratio and pivot criterion to decide the pivot row/constraint. Compute and compare ratios of right hand side constants to the coefficients of x_2 :

$$c_1 : \frac{1}{2}x_1 + 2x_2 + x_3 + x_4 = 24 \qquad \text{ratio: } \frac{24}{2} = 12$$

$$c_2 : x_1 + 2x_2 + 4x_3 + x_5 = 60 \qquad \text{ratio: } \frac{60}{2} = 30$$

So we **select the smallest ratio** 12. So pivot on c_2 and apply normalization/elimination steps. So we can divide c_1 by 2 to normalize the coefficient of x_2 to 1. Then, we can subtract c_1 from c_2 :

$$c'_1 = \frac{1}{2}c_1 : \qquad \frac{1}{4}x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 12$$

$$c'_2 = c_2 - c_1 : \qquad \frac{1}{2}x_1 + 0 + 3x_3 - x_4 + x_5 = 36$$

This is in a canonical form for the new basic variables x_2 and x_5 (x_2 replaced x_4). To complete the canonical form, we need to eliminate x_2 from the objective function. First,

$$x_2 = 12 - \frac{1}{4}x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

from c'_1 . We can substitute this into the objective function:

$$\begin{aligned} z' &= 6x_1 + 14x_2 + 13x_3 + 0x_4 + 0x_5 \\ &= 6x_1 + 14 \left(12 - \frac{1}{4}x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \right) + 13x_3 + 0x_4 + 0x_5 \\ &= 168 + \frac{5}{2}x_1 + 6x_3 - 7x_4 + 0x_2 + 0x_5 \end{aligned}$$

Now all conditions of canonical form are satisfied.

We can take our nonbasic variables x_1, x_3, x_4 and set them to 0. And our basic variables are x_2 and x_5 , which we can solve for to get our new BFS.

$$x_2 = 12 \qquad x_5 = 36$$

So our BFS is

$$\bar{x} = [0 \quad 12 \quad 0 \quad 0 \quad 36]$$

$$z' = 168$$

This is an improvement from the previous $z = 0$. However, this is still not an improvement because we have nonbasic variables in the objective function with positive coefficients.

The next variable we will convert to basic variable will be x_3 , because it has the biggest positive coefficient in the objective function z' . We can apply the pivoting criterion to the constraints c'_1 and c'_2

$$c'_1 : \frac{1}{4}x_1 + x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + 0x_5 = 12 \quad \text{ratio: } \frac{12}{\frac{1}{2}} = 24$$

$$c'_2 : \frac{1}{2}x_1 + 0x_2 + 3x_3 - x_4 + x_5 = 36 \quad \text{ratio: } \frac{36}{3} = 12$$

So, since 12 is the smallest ratio, we will use c'_2 as the pivot. In other words, we need to make x_3 's coefficient 1 in c'_2 , and eliminate x_3 in c'_1 .

$$c''_2 = \frac{1}{3}c'_2 : \quad \frac{1}{6}x_1 + 0x_2 + x_3 - \frac{1}{3}x_4 + \frac{1}{3}x_5 = 12$$

$$c''_1 = c'_1 - \frac{1}{2}c''_2 : \quad \frac{1}{6}x_1 + x_2 + 0x_3 + \frac{2}{3}x_4 - \frac{1}{6}x_5 = 6$$

Using the coefficients, we can describe them like this to see everything together

$$c''_1 : \left[\frac{1}{6} \quad 1 \quad 0 \quad \frac{2}{3} \quad -\frac{1}{6} \quad 6 \right]$$

$$c''_2 : \left[\frac{1}{6} \quad 0 \quad 1 \quad -\frac{1}{3} \quad \frac{1}{3} \quad 12 \right]$$

To find the values of our basic variables x_2 and x_3 , we can set the non basic variables to 0:

$$x_2 = 6$$

$$x_3 = 12$$

We can write x_3 in terms of the other variables in c''_2 :

$$x_3 = 12 - \frac{1}{6}x_1 + \frac{1}{3}x_4 - \frac{1}{3}x_5$$

We can find the next objective function z'' by plugging in x_2 into the older z' :

$$z'' = 168 + \frac{5}{2}x_1 + 6 \left(12 - \frac{1}{6}x_1 + \frac{1}{3}x_4 - \frac{1}{3}x_5 \right) - 7x_4$$

$$= 240 + \frac{3}{2}x_1 - 5x_4 - 2x_5 + 0x_2 + 0x_3$$

So our new BFS is

$$\bar{x} = [0 \quad 6 \quad 12 \quad 0 \quad 0]$$

$$z'' = 240$$

This is an improved value of $z'' = 240$. As we can see, we have one more step to find the optimal solution because we still have one variable, x_1 with a positive coefficient in the objective function.

2.3 Simplex Tableau

Definition 7 (Extended canonical form of LPP). Give the objective function of a maximization LPP, say

$$z = z_0 + c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in n decision variables, rewrite it so that the constant term z_0 is the right hand side term:

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = z_0$$

with variable z , or

$$(-z) + c_1x_1 + c_2x_2 + \cdots + c_nx_n = -z_0$$

with variable $-z$.

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Either version will work for a simplex tableau method, noting that the second version we try to make the coefficients c_1, c_2, \dots, c_n non positive gradually using improvement step.

A typical simplex tableau will look like a table, for example it may start out like this, assuming that x_1, x_2 are nonbasic variables, and x_3, x_4, x_5 are basic variables:

Basic variables	Current Values	Coefficients of variables	Constraints/constraint operations	Ratio test
x_3	b_1			C_1
x_4	b_2			C_2
x_5	b_3			C_3
$(-z)$	$-z_0$	c_1, c_2, c_3, c_4, c_5		C_4

This ends up being a more organized way of keeping track of our coefficients while we apply improvement procedures. We want to make all the coefficients in the last row non-positive. We can revise our BFS gradually (changing basic variables in each tableau). The resulting value of z_0 is the optimal value.

Example 12. Maximize

$$z = x_1 + 2x_2$$

subject to the constraints

$$\begin{aligned} x_1 - 2x_2 &\leq 3 \\ x_1 + x_2 &\leq 3x_1 && \geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Use two slack variables, x_3 and x_4 to get the constraints:

$$\begin{aligned} c_1 : x_1 - 2x_2 + x_3 &= 3 \\ c_2 : x_1 + x_2 + x_4 &= 3 \end{aligned}$$

Next, the z equation can be rewritten as

$$c_3 : (-z) + x_1 + 2x_2 + 0x_3 + 0x_4 = 0$$

We have basic variables x_3 and x_4 . The rest of the variables are our non basic variables x_1 and x_2 . Our BFS can be obtained by setting our non basic variables to 0:

$$(x_1, x_2, x_3, x_4) = (0, 0, 3, 3)$$

Now we can set up our first tableau:

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	3	1 -2 1 0	C_1	
x_4	3	1 1 0 1	C_2	
$(-z)$	0	1 2 0 0	C_3	

We can see that this is not optimal because the coefficients in the last row are still positive. We can start with a positive coefficient in that row. Since we are using the simplex procedure, we will select the largest positive coefficient to determine the entering variable. So we will select x_2 because it has a coefficient of 2, to be the next basic variable (to replace x_3 or x_4). So we can do the ratio test to see which column to pivot on (ratio of current value over coefficients of x_2 , ignore negative ratios).

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	3	1 -2 1 0	C_1	$\frac{3}{-2} = -2$
x_4	3	1 1 0 1	C_2	$\frac{3}{1} = 3$
$(-z)$	0	1 2 0 0	C_3	

So we pick the C_2 row to pivot on, because it has the smallest positive ratio. Note that x_2 in C_2 already has a coefficient of 1, so it is already normalized. We can use elimination to create a 0 in the x_2 column of C_1 :

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	9	3 0 1 2	$C'_1 = C_1 + 2C_2$	
x_2	3	1 1 0 1	$C'_2 = C_2$	
$(-z)$	-6	-1 0 0 -2	$C'_3 = C_3 - 2C_2$	

Now, since we have all non positive coefficients in the last row, we have reached an optimal value. Our optimal BFS is

$$(x_1, x_2, x_3, x_4) = (0, 3, 9, 0)$$

Since x_1 and x_2 are our original decision variables, we really only care about them. So our optimal value of $z = 6$ happens when $x_1 = 0$ and $x_2 = 3$.

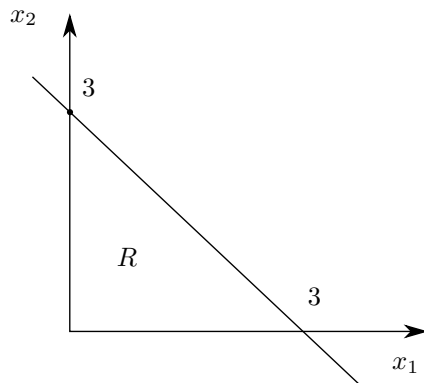


Figure 2.2: The feasible region, with the optimal solution at the point $(0, 3)$.

Example 13. Maximize

$$z = 2x_1 + x_2$$

subject to the constraints

$$\begin{aligned} x_2 &\leq 10 \\ 12x_1 + 3x_2 &\leq 6 \\ -3x_1 + x_2 &\leq 7 \end{aligned}$$

and sign constraints. We can introduce the slack variables x_3, x_4, x_5 for a canonical form:

$$\begin{aligned} C_1 : x_2 + x_3 &= 10 \\ C_2 : 12x_1 + 3x_2 + x_4 &= 6 \\ C_3 : -3x_1 + x_2 + x_5 &= 7 \end{aligned}$$

We can rewrite the z equation in this extended canonical form:

$$C_4 : (-z) + 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 = 0$$

So we have basic variables x_3, x_4, x_5 and non basic variables x_1, x_2 . So our first BFS is:

$$(x_1, x_2, x_3, x_4, x_5) = (0, 0, 10, 6, 7)$$

with a corresponding $z = 0$.

Lets translate the information to a tableau:

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	10	0 1 1 0 0	C_1	
x_4	6	12 3 0 1 0	C_2	
x_5	7	-3 1 0 0 1	C_3	
$(-z)$	0	2 1 0 0 0		

So, we know it is not optimal because we have positive coefficients in the last row. We can start with the first column (representing x_1), because it has the largest positive coefficient in the last row, 2. So x_1 is our entering variable. So lets look at ratios:

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	10	0 1 1 0 0	C_1	$\frac{10}{0}$, undefined
x_4	6	12 3 0 1 0	C_2	$\frac{6}{12} = \frac{1}{2}$
x_5	7	-3 1 0 0 1	C_3	$\frac{7}{-3}$
$(-z)$	0	2 1 0 0 0		

So we have one option for the pivot, the x_4 row. First, we need to normalize the x_4 row to create a 1 in the x_1 column. The new basic variable can be swapped out for x_1 , and then we will create 0s in the x_1 column using the normalized C_2 .

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	10	0 1 1 0 0	$C'_1 = C_1$	10
x_1	$\frac{1}{2}$	$1 \frac{1}{4} 0 \frac{1}{12} 0$	$C'_2 = \frac{1}{12}C_2$	2
x_5	$\frac{17}{2}$	$0 \frac{7}{4} 0 \frac{1}{4} 1$	$C'_3 = C_3 + 3C'_2$	$\frac{34}{7}$
$(-z)$	-1	$0 \frac{1}{2} 0 -\frac{1}{6} 0$	$C'_4 = C_4 - 2C'_2$	

And we have a new entering variable x_2 , because it still has a positive coefficient in the last row.

Procedure for minimzation LPP:

- Consider minimizing $z = z_0 + a_1x_1 + a_2x_2 + \dots + a_nx_n$ subject to some constraints and sign constraints.

- It is equivalent to maximizing $-z = -z_0 - a_1x_1 - a_2x_2 - \dots - a_nx_n$ subject to the same constraints.
- Let $z' = -z$, $z'_0 = -z_0$, $a'_k = -a_k$ for each k
- So, we get the equivalent maximization problem with:

$$\begin{aligned} -z_0 &= z - a_1x_1 - a_2x_2 - \dots - a_nx_n \\ (-z') + a'_1x_1 + a'_2x_2 + \dots + a'_nx_n &= -z'_0 \end{aligned}$$

Example 14. Minimize $z = 2 + x_1 - x_2$ subject to

$$2x_1 + x_2 \leq 2 \qquad x_1, x_2 \geq 0$$

Since we are minimizing, we want to keep x_1 small and x_2 large. See Figure ??.

Check that the optimal value occurs at the corner $(x_1, x_2) = (0, 2)$ with a corresponding $z = 0$ value. The new constraint with the slack variable x_3 is:

$$c_1 : x_1 + x_2 + x_3 = 2$$

with a corresponding objective function

$$\begin{aligned} c_2 : -z &= -2 - x_1 + x_2 \\ 2 &= z - x_1 + x_2 \end{aligned}$$

which we can put in canonical form:

$$z - x_1 + x_2 + 0x_3 = 2$$

So, our basic variable is x_3 , and our non basic variable x_1 and x_2 . Our initial BFS is

$$(x_1, x_2, x_3) = (0, 0, 2)$$

So lets use a simplex tableau to represent this information.

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_3	2	1 1 1	c_1	
z	2	-1 1 0	c_2	

We can tell that it is not optimal because x_2 has a positive coefficient in the objective function. x_2 will be our entering variable. Since we only have 1 basic variable, the ratio of that variable will be our pivot: $\frac{2}{1} = 2$.

Basic variables	Current Values	Coefficients of variables	Constraints	Ratio test
x_2	2	1 1 1	$c'_1 = c_1$	
z	0	-2 0 -1	$c'_2 = c_2 - c_1$	

Now that we have all negative values in our objective function's coefficients, we have obtained the optimal minimal value. So our BFS is

$$(x_1, x_2, x_3) = (0, 2, 0)$$

with a corresponding $z = 0$ value. This can be corroborated via the corner check we did in the beginning.

2.4 Big-M method

Handling “ \geq ” and “=” constraints If a “ \geq ” constraint is present with a non negative right hand side constant, then we apply surplus (excess) and artificial variables. For example,

$$\begin{aligned} 2x + y &\geq 3 \\ 2x + y - s &= 3 \end{aligned}$$

So what is our basic variable? We can choose one of the decision variables, x or y , by eliminating it in other constraints and objective function, or add another non negative variable called an artificial variable (with a +1 coefficient).

$$2x + y - s + a = 3$$

but make a to be 0 via a simplex procedure at the end. For pure “=” constraints, we only add artificial variables.

Example 15. Minimize $z = 2x_1 + x_2$ subject to the constraints

$$\begin{aligned} x_2 &\leq 10 \\ -3x_1 + x_2 &\leq 7 \end{aligned}$$

with sign constraints $x_1, x_2 \geq 0$.

$$\begin{aligned} c_1 : x_2 + x_3 &= 10 \\ c_2 : -3x_1 + x_2 - x_4 + x_5 &= 7 \end{aligned}$$

So we have x_4 as our surplus variable, and x_5 as our artificial variable. Our basic variables are x_3 and x_5 , and our non basic variables are x_1, x_2, x_4 . Our objective function is

$$z = 2x_1 + x_2 + 0(x_3 + x_4) + ?x_5$$

We want to penalize positive values of x_5 in any BFS so that the algorithm forces x_5 to be 0 in future BFS values. So, we can add Mx_5 into the z equation where M is a very large positive constant. (This is called the Big- M approach.)

$$z = 2x_1 + x_2 + 0(x_3 + x_4) + Mx_5$$

where $x_5 = 7 + 3x_1 - x_2 + x_4$. Then, in a corresponding max LPP canonical form:

$$\begin{aligned} 0 &= z - 2x_1 - x_2 - Mx_5 \\ &= z - 2x_1 - x_2 - M(7 + 3x_1 - x_2 + x_4) \\ &= z - (3M + 2)x_1 + (M - 1)x_2 - Mx_4 - 7M \end{aligned}$$

So, c_3 for z becomes

$$z - (3M + 2)x_1 + (M - 1)x_2 - Mx_4 = 7M$$

Lets describe this in a simplex tableau:

Basic variables	Current Values	x_1	x_2	x_3	x_4	x_5	Constraints	Ratio test
x_3	10	0	1	1	0	0	c_1	$\frac{10}{1} = 10$
x_5	7	-3	1	0	-1	1	c_2	$\frac{7}{1} = 7$
z	$7M$	$-3M - 2$	$M - 1$	0	$-M$	0	c_3	

Basic variables	Current Values	x_1	x_2	x_3	x_4	x_5	Constraints	Ratio test
x_3	3	3	0	1	1	-1	$c'_1 = c_2 - c_1$	
x_2	7	-3	1	0	-1	1	$c'_2 = c_2$	
z	7	-5	0	0	-1	$1 - M$	$c'_3 = c_3 - (M - 1)c_2$	

Consider a maximization LPP with an objective function z , and a constraint with a “ \geq ” or “ $=$ ”.

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For example,

$$c : 5x - 3y \geq 2$$

As usual, we integrate excess (surplus) and artificial variables:

$$5x - 3y - e + a = 2$$

where a is a candidate to be a basic variable.

We want the value of a to be zero in a subsequent BFS. So, if $a > 0$ in a BFS, we penalize this by adding $(-Ma)$ to z , where M is a large positive number. Then, we can maximize

$$z' = z - Ma$$

For multiple constraints that involve artificial variables, say a_1, a_2, \dots, a_r , we use

$$z - M(a_1 + a_2 + \dots + a_r)$$

in a maximization LPP. Similarly, for a minimization LPP we use

$$z + M(a_1 + a_2 + \dots + a_r)$$

We try to obtain an optimal solution to the extended LPP with non negative artificial variables. If all artificial variables are all 0, $a_k = 0$ for each k , then the resulting BFS is also the optimal solution to the original problem.

Example 16. Maximize

$$z = 3x_1 + 4x_2$$

subject to the constraints

$$x_1 + x_2 \leq 3$$

$$x_1 - 2x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Recall that the feasible set for this problem is empty (the constraints do not intersect).

$$c_1 : x_1 + x_2 + x_3 = 3$$

$$c_2 : x_1 - 2x_2 - x_4 + x_5 = 4$$

where x_3 is a slack variable, x_4 is an excess variable, and x_5 is an artificial variable. We want x_5 to be zero eventually in an optimal solution.

Our extended LPP is maximizing

$$z' = z + 0(x_3 + x_4) - Mx_5$$

where our basic variables are x_3 and x_5 . We can solve for x_5 :

$$x_5 = 4 - x_1 + 2x_2 + x_4$$

We write z' as

$$\begin{aligned} z' &= 3x_1 + 4x_2 - M(4 - x_1 + 2x_2 + x_4) \\ 4M &= (-z') + (3 + M)x_1 + (4 - 2M)x_2 - Mx_4 \end{aligned}$$

which is canonical form in simplex tableau.

BV	Value	x_1	x_2	x_3	x_4	x_5	Constraints	Ratio test
x_3	3	1	1	1	0	0	c_1	$\frac{3}{1} = 3$
x_5	4	1	-2	0	-1	1	c_2	$\frac{4}{1} = 4$
$(-z')$	$4M$	$3 + M$	$4 - 2M$	0	$-M$	0	c_3	

where x_1 is our entering variable, with the smallest ratio being 3, so we pivot on x_3 .

BV	Value	x_1	x_2	x_3	x_4	x_5	Constraints	Ratio test
x_1	3	1	1	1	0	0	$c'_1 = c_1$	
x_5	1	0	-3	-1	-1	1	$c'_2 = c_2 - c_1$	
$(-z')$	$M - 9$	0	$1 - 2M$	$-M - 3$	$-M$	0	$c'_3 = c_3 - (3 + M)c_1$	

This extended problem has an optimal solution $x_1 = 3$, $x_5 = 1$ and $x_k = 0$ for other k values, for large $M > 0$. However, this solution involves $x_5 = 1$ (which is positive, where we were trying to get it to be 0). In conclusion, the original problem does not have a solution.

Definition 8. If there is a constraint where the right hand side constraint is 0, we say that the LPP is degenerate.

Example 17. Minimize

$$z = x + 2y - 2$$

subject to

$$\begin{aligned} x &\leq 10 \\ -2x + 5y &\leq 0 \\ 2x - 5y &\geq 0 \\ 2x + 5y &\geq 20 \end{aligned}$$

In canonical form this would be

$$\begin{aligned} c_1 : x + s_1 &= 10 \\ c_2 : -2x + 5y + s_2 &= 0 \\ c_3 : 2x + 5y - e_1 + a &= 20 \end{aligned}$$

So

$$a = (20 - 2x - 5y + e_1)$$

and

$$c_4 : z' = x + 2y - 2 + Ma$$

We can then describe z' in a maximization LPP format for the initial simplex tableau. Our basic variables are s_1, s_2 and a . Our non basic variables are $s_1 = 10$. Our initial BFS is

$$(x, y, s_1, s_2, s_3, a) = (0, 0, 10, 0, 0, 20)$$

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