MATH100
Applied Linear Algebra

Zed Chance

Spring 2021

Contents

1 Vectors

> **Definition 1** (Vectors)**.** Vectors are directed line segments, they have both magnitude and direction. They exist in a "space," such as the plane $\mathbb{R}^2$, ordinary space $\mathbb{R}^3$, or an n-dimensional space $\mathbb{R}^n$.
>
> - In $\mathbb{R}^3$, the vector v can be represented by its components as $v = [v_1, v_2, v_3]$.
> - v can also be represented as a line segment with an arrowhead pointing in the direction of v.

Jan 23

Properties Vectors can be combined to form new vectors. Whether we are combining our vectors algebraically (manipulating their components) or geometrically (manipulating their graphs), the following **properties** apply: Let u, v, and w be vectors, and c and d be real numbers, then

$u + v = v + u$	commutative
$(u + v) + w = v + (u + w)$	associative
$c(du) = (cd)u$	associative
$u + 0 = u$	additive identity
$u + (-u) = 0$	additive inverse
$c(u + v) = cu + cv$	distributive
$(c + d)u = cu + du$	distributive
$1u = u$	multiplicative identity

Jan 27

Representing vectors Row vector:

$$\bar{V} = [2, 3]$$

Column vector:

$$\bar{V} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

> **Note 1.** Vectors u and v are equivalent if they have the same length and direction.

1.1 Vector properties

Let $\bar{u} = [1, 2]$ and $\bar{v} = [3, 1]$.

$$\begin{aligned} \bar{u} + \bar{v} &= [1, 2] + [3, 1] \\ &= [1 + 3, 2 + 1] \\ &= [4, 3] \end{aligned}$$

Geometrically, this is the "tip to tail" method. Any two vectors define a parallelogram.

Let $\bar{u} = [1, 2]$, and think about $\bar{u} + \bar{u}$.

$$\begin{aligned} \bar{u} + \bar{u} &= [1, 2] + [1, 2] \\ &= [2, 4] \\ 2\bar{u} &= 2[1, 2] \\ &= [2, 4] \end{aligned}$$

Also think about multiplying $\bar{u}$ by -1:

$$\begin{aligned} (-1)\bar{u} &= (-1)[1, 2] \\ &= [-1, -2] \end{aligned}$$

This points the vector in the opposite direction, which is considered "antiparallel". So if the scalar in the multiplication is a negative number, it will point the vector in the other direction (as well as being scaled).

Definition 2 (Scalar multiplication)**.** For constant c and $\bar{V} = [v_1, v_2, v_3]$, then

$$c\bar{V} = [cv_1, cv_2, cv_3]$$

Definition 3 (Vector subtraction)**.**

$$\bar{u} - \bar{v} = \bar{u} + (-\bar{v})$$

So with our existing vectors:

$$\begin{aligned} \bar{u} - \bar{v} &= \bar{u} + (-\bar{v}) \\ &= [1, 2] + [-3, -1] \\ &= [-2, 1] \end{aligned}$$

The sum and difference is the diagonals of the parallelogram created by adding the vectors.

Note 2. Vector addition is commutative, but vector subtraction is not (it is anticommutative).

$$\begin{aligned} \bar{v} - \bar{u} &= [3, 1] + [-1, -2] \\ &= [2, -1] \end{aligned}$$

Note 3. All of these properties hold true in all dimensions: $\mathbb{R}^n$.

Concerning the **additive identity**: In $\mathbb{R}^3$ the "zero vector" is $\bar{0} = [0,0,0]$.

How to represent the length of a vector:

$$\begin{aligned} \bar{u} &= [1,2] \\ &= \sqrt{1^2+2^2} \\ &= \sqrt{5} \\ ||\bar{u}|| &= \sqrt{5} \end{aligned}$$

We use the double bars to represent the length of a vector.

Jan 29

1.2 Linear combinations and coordinates

Definition 4. $\bar{v}$ is a linear combination of a set of vectors, $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$, if $\bar{v} = c_1\bar{v}_1, \ldots, c_k, \bar{v}_k$ for scalars c_i.

Example 1. See *Handout 1.*

Definition 5 (Standard Basis Vectors and Standard Coordinates)**.** In $\mathbb{R}^2 : \bar{e}_1 = [1,0], \bar{e}_2 = [0,1]$, these are the standard basis vectors.

Then $\bar{v} = [v_1, v_2]$,

and the standard coordinates of $\bar{v}$ are v_1, v_2.

1.3 Dot Product

Feb 01

Definition 6 (Dot Product)**.** If $\bar{u} = [u_1, u_2, \ldots, u_n]$, $\bar{v} = [v_1, v_2, \ldots, v_n]$, then the **dot product** of $\bar{u}$ with $\bar{v}$ is

$$\bar{u} \cdot \bar{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Example 2.

$$\begin{aligned} [2,-1,7] \cdot [3,5,-2] &= (2)(3) + (-1)(5) + (7)(-2) \\ &= 6 - 5 - 14 \\ &= -13 \end{aligned}$$

Properties of dot products (scalar products) Let $\bar{u}, \bar{v}, \bar{w}$ be vectors, and c be a scalar, then

$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$	commutative
$\bar{u} \cdot (\bar{v} + \bar{w}) = (\bar{u} \cdot \bar{v}) + (\bar{u} \cdot \bar{w})$	distributive
$(c\bar{u}) \cdot \bar{v} = c(\bar{u} \cdot \bar{v})$	
$\bar{0} \cdot \bar{v} = 0$	
$\bar{v} \cdot \bar{v} = v_1^2 + v_2^2 + \cdots + v_n^2$	

Length In $\mathbb{R}^2$: $||\bar{v}|| = \sqrt{v_1^2 + v_2^2}$

In general: $||\bar{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

Example 3. If $\bar{v} = [2, -1, 7]$, then the length is

$$\begin{aligned} ||\bar{v}|| &= \sqrt{2^2 + (-1)^2 + 7^2} \\ &= \sqrt{4 + 1 + 49} \\ &= 3\sqrt{6} \end{aligned}$$

Note 4.

$$||\bar{v}|| = \sqrt{\bar{v} \cdot \bar{v}}$$

Definition 7. A vector of length 1 is called **unit vector**. For any vector $\bar{v} \neq \bar{0}$: $\frac{\bar{v}}{||\bar{v}||}$ is a unit vector in the same direction as $\bar{v}$.

Note 5.

$$\bar{v}\left(\frac{||\bar{v}||}{||\bar{v}||}\right) = ||\bar{v}||\left(\frac{\bar{v}}{||\bar{v}||}\right)$$

Example 4. In $\mathbb{R}^2 : \bar{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, these are unit vectors.

Important inequalities

- Triangle inequality

  The triangle created by the parallelogram of a vector addition, the length of any one side cannot be greater than the sum of the other two sides.

  $$||\bar{u} + \bar{v}|| \leq ||\bar{u}|| + ||\bar{v}||$$

- Cauchy-Schwarz inequality

  $$|\bar{u} \cdot \bar{v}| \leq ||\bar{u}||\,||\bar{v}||$$

  Proof by the Law of Cosines:

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\theta \\
||\bar{u} - \bar{v}||^2 &= ||\bar{u}||^2 + ||\bar{v}||^2 - 2||\bar{u}||\,||\bar{v}||\cos\theta \\
(\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v}) &= \\
||\bar{u}||^2 - 2(\bar{u} \cdot \bar{v}) + ||\bar{v}||^2 &= \\
||\bar{u}||^2 - 2(\bar{u} \cdot \bar{v}) + ||\bar{v}||^2 &= ||\bar{u}||^2 + ||\bar{v}||^2 - 2||\bar{u}||\,||\bar{v}||\cos\theta \\
\bar{u} \cdot \bar{v} &= ||\bar{u}||\,||\bar{v}||\cos\theta \\
|\bar{u} \cdot \bar{v}| &= ||\bar{u}||\,||\bar{v}||\,|\cos\theta| \\
|\bar{u} \cdot \bar{v}| &\leq ||\bar{u}||\,||\bar{v}||.
\end{aligned}
$$

Angle θ between vectors $\bar{u}$ and $\bar{v}$ Excluding the zero vector:

Let $0 \leq \theta \leq \pi$,

$$\cos\theta = \frac{\bar{u} \cdot \bar{v}}{||\bar{u}||\,||\bar{v}||}$$

So, $\theta = \cos^{-1}\left(\frac{\bar{u} \cdot \bar{v}}{||\bar{u}||\,||\bar{v}||}\right)$

Note 6. If $\bar{u}, \bar{v} \neq 0$, then $\theta = \frac{\pi}{2}$, if and only if $\bar{u} \cdot \bar{v} = 0$.

$$\bar{u} \perp \bar{v}, \text{ iff } \bar{u} \cdot \bar{v} = 0$$

Feb 03

1.4 Distance between vectors

Definition 8. The distance between two vectors is the distance between their tips.

If $\bar{u} = [u_1, u_2]$ and $\bar{v} = [v_1, v_2]$, then

$$
\begin{aligned}
d(\bar{u}, \bar{v}) &= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \\
&= ||\bar{u} - \bar{v}|| \\
&= d(\bar{v}, \bar{u}) \\
&= ||\bar{v} - \bar{u}||
\end{aligned}
$$

Example 5. In $\mathbb{R}^3$:

For $\bar{u} = [2, -1, 7]$ and $\bar{v} = [3, 5, -2]$:

Find the distance:

$$
\begin{aligned}
d(\bar{u},\bar{v}) &= ||\bar{u}-\bar{v}|| \\
&= ||[(2-3),(-1-5),(7+2)]|| \\
&= ||[-1,-6,9]|| \\
&= \sqrt{(-1)^2+(-6)^2+9^2} \\
&= \sqrt{1+36+81} \\
&= \sqrt{118}
\end{aligned}
$$

1.5 Projections

Definition 9. Let $proj_{\bar{u}}\bar{v}$ be the vector projection of $\bar{v}$ onto $\bar{u}$, then the signed length of $proj_{\bar{u}}\bar{v}$ is given by

$$
\begin{aligned}
||\bar{v}||\cos\theta &= ||\bar{v}||\frac{\bar{u}\cdot\bar{v}}{||\bar{v}||||\bar{u}||} \\
&= \frac{\bar{v}\cdot\bar{u}}{||\bar{u}||}
\end{aligned}
$$

So,

$$
\begin{aligned}
proj_{\bar{u}}\bar{v} &= \left(\frac{\bar{v}\cdot\bar{u}}{||\bar{u}||}\right)\frac{\bar{u}}{||\bar{u}||} \\
&= \left(\frac{\bar{v}\cdot\bar{u}}{\bar{u}\cdot\bar{u}}\right)\bar{u}
\end{aligned}
$$

Note 7. Recall, $\bar{u}\cdot\bar{u} = ||\bar{u}||^2$

Note 8. Remember, $\frac{\bar{u}}{||\bar{u}||}$ is the unit vector.

$$
\frac{\bar{v}\cdot\bar{u}}{||\bar{u}||} = \bar{v}\frac{\bar{u}}{||\bar{u}||}
$$

Example 6. For $\bar{u} = [2,1,-2]$ and $\bar{v} = [3,0,8]$, find the projection of $\bar{v}$ onto $\bar{u}$:

$$\begin{aligned} proj_{\bar{u}}\bar{v} &= \frac{[3,0,8]\cdot[2,1,-2]}{[2,1,-2]\cdot[2,1,-2]}[2,1,-2] \\ &= \frac{6+0-16}{4+1+4}[2,1,-2] \\ &= \frac{-10}{9}[2,1,-2] \end{aligned}$$

Since the coefficient is negative, the angle between the two vectors is more than 90 degrees.

1.6 Lines and planes

See *Handout 2*

Comparing vector and parametric forms:

$$\begin{aligned} \bar{x} &= \bar{p} + t\bar{d} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + t\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ x &= p_1 + td_1 \\ y &= p_2 + td_2 \end{aligned}$$

The solution to this is the line l.

Comparing the normal and general forms:

Let $\bar{n} = \begin{bmatrix} a \\ b \end{bmatrix}$:

$$\begin{aligned} \bar{n}\cdot\bar{x} &= \bar{n}\cdot\bar{p} \\ \begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ ax + by &= ap_1 + bp_2 \\ ax + by &= c \end{aligned}$$

Remember, $ap_1 + bp_2$ are constants, so we can call them c.

Example 7. See *Handout 1.*

Find an equation for that line that passes through the point (-3, 2) and is parallel to the vector [2, 1].

1. Vector form

$$\begin{aligned}\bar{x} &= \bar{p} + t\bar{d} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -3 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}\end{aligned}$$

2. Parametric form

$$\begin{aligned}x &= -3 + 2t \\ y &= 2 + t\end{aligned}$$

Feb 05

Example 8. *Cont from previous example*

3. General form

$$\begin{aligned}t = \frac{x+3}{2} &= \frac{y-2}{1} \\ x + 3 &= 2y - 4 \\ x - 2y &= -7\end{aligned}$$

4. Normal form

$$\begin{aligned}\bar{n} \cdot \bar{x} &= \bar{n} \cdot \bar{p} \\ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix}\end{aligned}$$

Making sense of the normal form *See handout 2's graph*

1. Note that $\bar{x} - \bar{p}$ is parallel to the line l.
2. Also, $\bar{n} \perp (\bar{x} - \bar{p})$ by definition of $\bar{n}$.
3. Then, $\bar{n} \cdot (\bar{x} - \bar{p}) = 0$ by a property of dot products.

   We can use the distributive property:

$$\begin{aligned}\bar{n} \cdot \bar{x} - \bar{n} \cdot \bar{p} &= 0 \\ \bar{n} \cdot \bar{x} &= \bar{n} \cdot \bar{p}\end{aligned}$$

1.7 Lines in $\mathbb{R}^3$

See handout 2

Vector form

$$\bar{x} = \bar{p} + t\bar{d}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + t \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Parametric form

$$x = p_1 + td_1$$
$$y = p_2 + td_2$$
$$z = p_3 + td_3$$

Example 9. *See handout 2*

Find vector and parametric forms for the equation for the line containing the points (2, 4, -3) and (3, -1, 1).

$$\bar{p_1} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$
$$\bar{p_2} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$
$$\bar{d} = \bar{p_2} - \bar{p_1}$$
$$= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$
$$\bar{d} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Pick one of the points for our point vector $\bar{p}$.

1. Vector form

$$\bar{x} = \bar{p} + t\bar{d}$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

2. Parametric form

$$
\begin{aligned}
x &= 2 + t \\
y &= 4 - 5t \\
z &= -3 + 4t
\end{aligned}
$$

1.8 Planes in $\mathbb{R}^3$

See handout 2

Normal form Let $\bar{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$
\begin{gathered}
\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p} \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}
\end{gathered}
$$

General form

$$
\begin{aligned}
&ax + by + cz = ap_1 + bp_2 + cp_3 \\
&ax + by + cz = d
\end{aligned}
$$

We can combine the constants on the right into one single constant, d.

Example 10. *See handout 2*

Find normal and general forms for the equation of the plane orthogonal to the vector [2,3,4] that passes through the point (2,4,-1).

Let $\bar{n} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$, and $\bar{p} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$. We can start off by putting this in the normal form.

1. Normal form

$$
\begin{gathered}
\bar{n} \cdot \bar{x} = \bar{n} \cdot \bar{p} \\
\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}
\end{gathered}
$$

2. General form

$$
\begin{aligned}
&2x + 3y + 4z = (2)(2) + (3)(4) + (4)(-1) \\
&2x + 3y + 4z = 12
\end{aligned}
$$

Example 11. *See handout 2*

Find a vector form for the plane in the previous example.

Let $\bar{u}, \bar{v}$ be in the plane. Then, $\bar{u} \perp \bar{n}$, $\bar{v} \perp \bar{n}$, so

$$\bar{u} \cdot \bar{n} = 0 \qquad \bar{v} \cdot \bar{n} = 0$$

And, $\bar{u}$ is not parallel to $\bar{v}$.

Let $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$, where $u_3 = 0$. Then,

$$\begin{aligned} \bar{n} \cdot \bar{u} &= 0 \\ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} &= 0 \\ 2u_1 + 3u_2 &= 0 \end{aligned}$$

Let $\bar{u} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$.

Let $\bar{v} = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}$, where $v_2 = 0$.

Then,

$$\begin{aligned} \bar{n} \cdot \bar{v} &= 0 \\ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix} &= 0 \\ 2v_1 + 4v_3 &= 0 \end{aligned}$$

Let $\bar{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

So the vector form is

$$\begin{aligned} \bar{x} &= \bar{p} + s\bar{u} + t\bar{v} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \end{aligned}$$

And our parametric form is

$$\begin{aligned} x &= 2 + 3s + 2t \\ y &= 4 - 2s \\ z &= -1 - t \end{aligned}$$

2 Systems of Linear Equations

Definition 10. A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form:

Feb 08

$$a_1x_2 + a_2x_2 + \cdots + a_nx_n = b$$

where the coefficients $a_1, \ldots, a_n$ and the constant term b is constant.

Definition 11. A finite set of linear equations is a **system of linear equations**. A **solution set** of a system of linear equations is the set of *all* solutions of the system. A system of linear equations is either "consistent" if it has a solution, or it is "inconsistent" if there is no such solution.

Theorem 1. A system of linear equations has **either**

1. A unique solution – consistent
2. Infinitely many solutions – consistent
3. No solution – inconsistent

Definition 12. Two linear systems are said to be **equivalent** if they have the same solution set.

Example 12. *See handout 3, problem 1*

$$\begin{aligned} 2x + y &= 8 \\ x - 3y &= -3 \\ y &= 2, x = 3 \end{aligned}$$

Example 13. *See handout 3, problem 2*

Example 14. *See handout 3, problem 1*

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$$

be the matrix of coefficients, and let

$$\bar{b} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$$

Be the vector of constants.

Then,

$$[A \mid \bar{b}] = \begin{bmatrix} 2 & 1 \mid 8 \\ 1 & -3 \mid -3 \end{bmatrix}$$

is called the augmented matrix.

2.1 Direct methods of solving systems

Example 15. For the system Feb 10

$$\begin{aligned} 2x - y &= 3 \\ x + 3y &= 5 \end{aligned}$$

The coefficient matrix A is

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

The constant vector $\bar{b}$ is

$$\bar{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

The augmented matrix is

$$[A \mid \bar{b}] = \begin{bmatrix} 2 & -1 & \mid & 3 \\ 1 & 3 & \mid & 5 \end{bmatrix}$$

Definition 13 (Row echelon form of a matrix)**.** *See handout 4*

2.2 Gaussian and Gauss-Jordan Elimination

To solve a system of linear equations using Guassian Elimination: Feb 12

- Write out an augmented matrix for the system of linear equations
- Use elementary row operations to reduce the matrix to row echelon form
- Write out a system of equations corresponding to the row echelon matrix
- Use back substitution to find solution(s), if any exist, to the new system of equations

To solve the system of linear equations using Gauss-Jordan Elimination: reduce the augmented matrix to reduced row echelon form

- Write out the system of equations corresponding to the reduced row echelon matrix
- Use back substitution to find solution(s), if any exist, to the new system of equations

Example 16. *See handout 5*

Feb 15

Example 17. *See handout 5*

2.3 Spanning Sets and Linear Independence

Feb 17

Definition 14 (Linear Combinations of Vectors)**.** Let $\bar{V}$ be a linear combination of the set of vectors $\bar{V}_1, \ldots, \bar{V}_k$, if you can write $\bar{V}$ as the sum of scalar multiples of the set of vectors,

$$\bar{V} = c_1\bar{V}_1 + \cdots + c_k\bar{V}_k$$

for constants $c_1, \ldots, c_k$.

Example 18. Is $\begin{bmatrix} 8 \\ -3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$?

Alternatively, does $\begin{bmatrix} 8 \\ -3 \end{bmatrix} = x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ have a solution?

Equivalently, does the following system have a solution?

$$\begin{aligned} 2x + y &= 8 \\ x - 3y &= -3 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 8 \\ 1 & -3 & -3 \end{bmatrix} = R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -3 & -3 \\ 2 & 1 & 8 \end{bmatrix}$$
$$= R_2 - 2R_1 \begin{bmatrix} 1 & -3 & -3 \\ 0 & 7 & 14 \end{bmatrix}$$
$$= \frac{1}{7}R_2 \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 2 \end{bmatrix}$$
$$= R_1 + 3R_2 \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

So,

$$x = 3 \qquad\qquad y = 2$$

Example 19. For what values a, b will $\begin{bmatrix} a \\ b \end{bmatrix}$ be a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$?

$$\begin{bmatrix} a \\ b \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

If and only if,

$$2x + y = a$$
$$x - 3y = b$$

So we can solve it using our augmented matrix:

$$\begin{bmatrix} 2 & 1 & a \\ 1 & -3 & b \end{bmatrix} = R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -3 & b \\ 2 & 1 & a \end{bmatrix}$$
$$= R_2 - 2R_1 \begin{bmatrix} 1 & -3 & b \\ 0 & 7 & a - 2b \end{bmatrix}$$
$$= \frac{1}{7}R_2 \begin{bmatrix} 1 & -3 & b \\ 0 & 1 & \frac{a-2b}{7} \end{bmatrix}$$
$$= R_1 + 3R_2 \begin{bmatrix} 1 & 0 & b + 3(\frac{a-2b}{7}) \\ 0 & 1 & \frac{a-2b}{7} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \frac{3a+b}{7} \\ 0 & 1 & \frac{a-2b}{7} \end{bmatrix}$$

So we have

$$x = \frac{3a+b}{7}$$
$$y = \frac{a-2b}{7}$$

So the answer is that any choice of a, b will work. We can say that $\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\-3\end{bmatrix}$ "span" the plane ($\mathbb{R}^2$).

Definition 15 (Spanning Sets)**.** If $S = \{\bar{V}_1, \ldots, \bar{V}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\bar{V}_1, \ldots, \bar{V}_k$ is called the **span** of $\bar{V}_1, \ldots, \bar{V}_k$, or

$$\text{span}(S)$$

If the $\text{span}(S) = \mathbb{R}^n$, then we say S is a **spanning set** of $\mathbb{R}^n$.

Example 20. Describe the span of S where

$$S = \{ \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-1\\1\\-3\end{bmatrix} \}$$

Another way to think about is it, what vectors $\begin{bmatrix}a\\b\\c\end{bmatrix}$ are in the span of S?

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = x + \begin{bmatrix}1\\0\\3\end{bmatrix} + y \begin{bmatrix}-1\\1\\-3\end{bmatrix}$$

So,

$$x - y = a$$
$$y = b$$
$$3x - 3y = c$$

So we can use our augmented matrix:

$$\begin{bmatrix} 1 & -1 & a \\ 0 & 1 & b \\ 3 & -3 & c \end{bmatrix} = R_3 - 3R_1 \begin{bmatrix} 1 & -1 & a \\ 0 & 1 & b \\ 0 & 0 & c-3a \end{bmatrix}$$
$$= R_1 + R_2 \begin{bmatrix} 1 & 0 & a+b \\ 0 & 1 & b \\ 0 & 0 & c-3a \end{bmatrix}$$

So this system only has solutions if $c - 3a = 0$ or $c = 3a$. So vectors of the form $\begin{bmatrix} a \\ b \\ 3a \end{bmatrix}$ form the span of S.

Note 9. Linear systems of the form

$$\begin{aligned} x - y &= a \\ y &= b \\ 3x - 3y &= 3a \end{aligned}$$

have solutions for a, b arbitrarily. a, b are the free variables, but the third variable must be 3 times a.

2.4 Linear Independence

Definition 16. A set of vectors $\bar{v}_1, \ldots, \bar{v}_k$ is **linearly dependent** if there are scalars $c_1, \ldots, c_k$ (not all zero), such that Feb 22

$$c_1\bar{v}_1 + \cdots c_k\bar{v}_k = \bar{0}$$

Otherwise, the set is **linearly independent**.

Example 21. Decide if the set $\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \}$ is linearly independent.

So this is asking if this is true:

$$c_1 + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for non-trivial constants.

This leads to an augmented matrix:

$$\begin{aligned}
\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} &= R_2 - 2R_1 \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \\
&= R_1 + R_2 \text{ and } R_3 - R_2 \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= (-1)R_2 \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

So we have

$$\begin{aligned} c_1 + 3c_3 &= 0 \\ c_2 - 3c_3 &= 0 \end{aligned}$$

So we can solve for c_3 :

$$\begin{aligned} c_1 &= -3c_3 \\ c_2 &= 2c_3 \end{aligned}$$

So c_3 is arbitrary, it doesn't have to be 0. So the answer has non-trivial solutions, therefore it is linearly dependent.

$$-3c_3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is called the "linear dependence relation."

Note 10. A matrix with all 0s in the rightmost column is called a **homogeneous system of equations**.

Theorem 2. Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

Example 22. Consider this set of vectors

$$S = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \}$$

We can tell without doing anything else that these vectors have to be dependent. They are in $\mathbb{R}^2$, but there are 3 vectors total. We are guaranteed that there is a non-trivial linear combination that will make the $\bar{0}$.

$$0\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

However, it does not guarantee that one of the vectors can be solved as a linear combination of the others:

$$\begin{bmatrix}1\\0\end{bmatrix}=c_1\begin{bmatrix}0\\1\end{bmatrix}+c_2\begin{bmatrix}0\\2\end{bmatrix}$$

has no solution.

Feb 24

Example 23. *See handout 6*

3 Matrices

See handout X

See handout 7 Mar 01

See handout 8 Mar 05

3.1 Subspaces of Matrices

Mar 10

Definition 17. Subspaces of $\mathbb{R}^n$: A collection S of vectors in $\mathbb{R}^n$ such that

1. The zero vector $\bar{0}$ is in S
2. If $\bar{u}, \bar{v}$ are both in S, then $\bar{u}+\bar{v}$ are in S
3. If $\bar{u}$ is in S, then any scalar multiple $c\bar{u}$ is in S.

If all are true, then S is a subspace in $\mathbb{R}^n$

You can combine 2 and 3 above as: If $\bar{u}_1, \ldots, \bar{u}_k$ are in S and $c_1, \ldots, c_k$ are scalars, then $c_1\bar{u}_1 + \cdots + c_k\bar{u}_k$ is in S. S is closed under linear combinations.

Theorem 3. Let $\bar{v}_1, \dots, \bar{v}_k$ be vectors in $\mathbb{R}^n$, then $S = \text{span}(\bar{v}_1, \dots, \bar{v}_k)$ is a subspace of $\mathbb{R}^n$.

Proof:

Recall that the span of a set of vectors $(\bar{v}_1, \dots, \bar{v}_k)$ is the set of all linear combinations of $\bar{v}_1, \dots, \bar{v}_k$.

1. $\bar{0} = 0\bar{v}_1 + \cdots + 0\bar{v}_k$, so $\bar{0}$ is in the span S.
2. Let $\bar{u} = c_1\bar{u}_1 + \cdots + c_k\bar{u}_k$, then by definition $\bar{u}$ is in the span S. Also, $\bar{v} = d_1\bar{v}_1 + \cdots + d_k\bar{v}_k$

$$\begin{aligned}\bar{u} + \bar{v} &= (c_1\bar{v}_1 + \cdots + c_k\bar{v}_k) + (d_1\bar{v}_1 + \cdots + d_k\bar{v}_k)\\ &= (c_1 + d_1)\bar{v}_1 + \cdots + (c_k + d_k)\bar{v}_k\end{aligned}$$

   So $\bar{u} + \bar{v}$ is in the span S.
3. If $\bar{u}$ is in S, then $c\bar{u} = c(c_1\bar{v}_1 + \cdots + c_k\bar{v}_k)$, then

$$\begin{aligned}c\bar{u} &= c(c_1\bar{v}_1 + \cdots + c_k\bar{v}_k)\\ &= cc_1\bar{v}_1 + \cdots + cc_k\bar{v}_k\end{aligned}$$

   So $c\bar{u}$ is in S.

Example 24. *See handout 11*

3.2 Nullspace

Example 25. *See handout 11* Mar 12

3.3 Column space

3.4 Linear transformations Mar 17

See handout 12 Mar 29

Definition 18. The map $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

1. $T\underbrace{(\bar{u} + \bar{v})}_{\mathbb{R}^n} = T(\underbrace{\bar{u}}_{\mathbb{R}^m}) + T(\underbrace{\bar{v}}_{\mathbb{R}^m})$
2. $T(c\bar{v}) = cT(\bar{v})$

   Alternatively: $T(c\bar{u} + d\bar{v}) = cT(\bar{u}) + dT(\bar{v})$

   For every $\bar{u}, \bar{v} \in \mathbb{R}^n$, and $c, d \neq 0$

Theorem 4. *See Theorem 3.30 in Poole*

Let A be an $m \times n$ matrix. Then the map defined by $A\bar{x}$ is a linear for $\bar{x} \in \mathbb{R}^n$

Proof:

$A(c\bar{u} + d\bar{v}) = cA\bar{u} + dA\bar{v}$ by properties of matrix multiplication.

We can write this as $T_A(\bar{x}) = A\bar{x}$

Theorem 5. *See Theorem 3.31 in Poole*

This is an important theorem!

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $m \times n$ matrix A such that $T = T_A$. Specifically let $\bar{e}_1, \dots, \bar{e}_n$ be the standard basis for $\mathbb{R}^n$.

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \end{bmatrix} \qquad \bar{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

So we can find A by:

$$A = [T(\bar{e}_1) \mid \dots \mid T(\bar{e}_n)]_{m \times n}$$

See notes for proof

$$\begin{aligned} \bar{x} &= \bar{x}_1\bar{e}_1 + \cdots + \bar{x}_n\bar{e}_n \\ T(\bar{x}) &= T(\bar{x}_1\bar{e}_1 + \cdots + \bar{x}_n\bar{e}_n \\ &= \bar{x}_1 T(\bar{e}_1) + \cdots + x_n T(\bar{e}_n) \\ &= \begin{bmatrix} T(\bar{e}_1) & \dots & T(\bar{e}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ T(\bar{x}) &= A\bar{x} \end{aligned}$$

Example 26. *See handout 14*

Apr 02

Example 27. *See handout 14, example 3*

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection of the vector $\bar{v}$ onto the line ℓ through the origin.

See notes for drawing

Show that T is a linear transformation.

Let $\hat{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ be a direction vector for ℓ, where $||\hat{d}|| = \sqrt{d_1^2 + d_2^2}$. Note that

$$\begin{aligned} T(\bar{v}) &= proj_{\hat{d}}\bar{v} \\ &= \left(\frac{\bar{v} \cdot \hat{d}}{||\hat{d}||^2} \right) \end{aligned}$$

So our strategy is to find $T(\bar{e}_1)$ and $T(\bar{e}_2)$.

$$\begin{aligned} T(\bar{e}_1) &= proj_{\hat{d}}\bar{e}_1 \\ &= \left(\frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}{||\hat{d}||^2} \right) \hat{d} \\ &= d_1 \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} d_1^2 \\ d_1 d_2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} T(\bar{e}_2) &= proj_{\hat{d}}\bar{e}_2 \\ &= \left(\frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}{1} \right) \hat{d} \\ &= d_2 \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} d_1 d_2 \\ d_2^2 \end{bmatrix} \end{aligned}$$

So the standard matrix of T is

$$A = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

So, the projection onto a line through the origin is a linear transformation.

Example 28. Special case:

Project $\bar{v}$, in the plane, onto the x-axis.

Let $\bar{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

We can drop this to the x-axis, and see that

$$T(\bar{v}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Let

$$\begin{aligned} \hat{d} &= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \bar{e}_1 \end{aligned}$$

We can use our standard matrix we found in the previous example. Note that $d_1 = 1$ and $d_2 = 0$.

$$\begin{aligned} A &= \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Example 29. Another special case:

Project $\bar{v}$, in the plane, onto the line $y = x$.

We can use our standard matrix we found in the previous example.

Let $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, and

$$\begin{aligned} \hat{d} &= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\ &= \begin{bmatrix} a \\ a \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} ||\hat{d}|| &= \sqrt{a^2 + a^2} \\ &= d\sqrt{2a^2 A} \\ &= |a| d\sqrt{2} \\ &= 1 \\ |a| &= \frac{1}{\sqrt{2}} \end{aligned}$$

Let

$$\hat{d} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

So

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} T(\bar{v}) &= A\bar{v} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} x+y \\ x+y \end{bmatrix} \\ &= \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix} \end{aligned}$$

Both components equal the average of the components in the original vector. This is important in statistics.

Example 30. Derive the formula for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

Recall the rotation matrix from *Example 1 on Handout 14* that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} T(\bar{v}) &= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} \end{aligned}$$

So, since the components of equal vectors are equal to each other:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha \\ \sin(\alpha+\beta) &= \sin\beta\cos\alpha + \cos\beta\sin\alpha \end{aligned}$$

However, usually the text book will rearrange this:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}$$

We can also find the difference of the angles by thinking about

$$\cos(\alpha - \beta) = \cos(\alpha + (-\beta))$$
$$\cos(-\beta) = \cos\beta$$
$$\sin(-\beta) = -\sin\beta$$

Apr 05

Example 31. Additional questions for *example 3 on Handout 14*:

Recall: $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T(\bar{v}) = A\bar{v}$.

1. What is the range of the projection?

   The line ℓ is the range. Since every vector gets projected onto ℓ, that is the range.

2. Is the line ℓ a subspace of $\mathbb{R}^2$?

   Yes! Recall that spaces need to include the zero vector $\bar{0}$.

   - The line ℓ contains the point $(0, 0)$.
   - If $\bar{u}||\ell$, then $\bar{u} + \bar{v}||\ell$.
   - If $\bar{u}||\ell$, then $c\bar{u}||\ell$.

3. Find a basis for the subspace ℓ.

$$\{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\}$$

$$\begin{aligned} \bar{u} &= a\hat{d} \\ \bar{v} &= b\hat{d} \\ \bar{u} + \bar{v} &= a\hat{d} + b\hat{d} \\ &= (a+b)\hat{d} \end{aligned}$$

   So, $(a + b)\hat{d}||\ell$.

$$\begin{aligned} c\bar{u} &= c(a\hat{d}) \\ &= (ca)\hat{d} \end{aligned}$$

   and, $(ca)\hat{d}||\ell$.

4. Describe the column space of A. Recall:

$$A = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_2^2 \end{bmatrix}$$

   The answer is the line ℓ.

$$A\bar{v} = \bar{b}$$

   A big takeaway: the range of T is the column space of A.

Lets find the column space of A, using the augmented matrix:

$$\begin{aligned}
[A \mid \bar{b}] &= \begin{bmatrix} d_1^2 & d_1d_2 & a \\ d_1d_2 & d_2^2 & b \end{bmatrix} \\
&= (d_2)R_1, (d_1)R_2 \begin{bmatrix} d_1^2d_2 & d_1d_2^2 & d_2a \\ d_1^2d_2 & d_1d_2^2 & d_1b \end{bmatrix} \\
&= R_2 - R_1 \begin{bmatrix} d_1^2d_2 & d_1d_2^2 & d_2a \\ 0 & 0 & d_1b - d_2a \end{bmatrix} \\
d_1b &= d_2a \\
b &= \frac{d_2}{d_1}a
\end{aligned}$$

Notice that this is a line through the origin with a slope of $m = \frac{d_2}{d_1}$, which is the line ℓ. So the column space of the matrix A is the same as the range of T. The basis for the column space is the same as the basis of the range of T.

$$Col(A) = \{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\}$$

5. What is the rank of A?

   The $Rank(A) = dim(Col(A)) = dim(Row(A))$, so the rank is 1.

6. Describe the null space of A. We are looking for $A\bar{v} = \bar{0}$.

   Any vector that is orthogonal to ℓ and passes through the origin will be projected to the zero vector. This is the line ℓ_2 where the slope is $m = -\frac{d_1}{d_2}$.

   So lets show this analytically:

$$\begin{aligned}
[A \mid \bar{0}] &= \begin{bmatrix} d_1^2 & d_1d_2 & 0 \\ d_1d_2 & d_2^2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} d_1^2d_2 & d_1d_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
d_1x + d_2y &= 0 \\
y &= -\frac{d_1}{d_2}x
\end{aligned}$$

7. Find a basis for the null space $Null(A)$.

$$\{\begin{bmatrix} d_2 \\ -d_1 \end{bmatrix}\}$$

Example 32. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ with a standard matrix A, and $S : \mathbb{R}^n \to \mathbb{R}^p$ with a standard matrix B.

$$\begin{aligned}
S(T(\bar{v})) &= S(B\bar{v}) \\
&= A(B\bar{v}) \\
&= (AB)\bar{v} \\
&= (S \circ T)(\bar{v})
\end{aligned}$$

3.5 Composition of linear transformations

Apr 07

Definition 19. If $S(\bar{u}) = A\bar{u}$ and $T(\bar{v}) = B\bar{v}$, then

$$\begin{aligned} S(T(\bar{v})) &= (S \circ T)(\bar{v}) \\ &= AB\bar{v} \end{aligned}$$

and AB is the standard matrix for this composition $(S \circ T)(\bar{v})$.

Example 33. Show that reflection in the plane about the x-axis is a linear transformation.

See notes for drawing

$T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$\begin{aligned} T(\bar{e}_1) &= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \\ &= \begin{bmatrix}1\\0\end{bmatrix} \\ &= \bar{e}_1 \end{aligned}$$

$$\begin{aligned} T(\bar{e}_1) &= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) \\ &= \begin{bmatrix}0\\-1\end{bmatrix} \\ &= -\bar{e}_2 \end{aligned}$$

So the standard matrix for T is

$$A = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Since we found a matrix that implements the transformation, that means that reflection about the x-axis must be linear.

Example 34. Let $F_x : \mathbb{R}^2 \to \mathbb{R}^2$ be a reflection about the x-axis. Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by θ. Find the standard matrix for $R_{60\,\mathrm{deg}}(F_x(\bar{v}))$.

See notes for drawing

$$\begin{aligned} R_{60\,\mathrm{deg}} &= \begin{bmatrix}\cos 60 & -sin60\\ sin60 & cos60\end{bmatrix} \\ F_x &= \begin{bmatrix}1 & 0\\ 0-1 & \end{bmatrix} \\ R_{60} \circ F_x &= \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{-2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix} \\ &= \begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\bar{v} \end{aligned}$$

This is the standard matrix for reflection about the x-axis followed by rotation of 60 degrees.

Example 35. Find the standard matrix that rotates by 60 degrees, then reflects about the x-axis. This is reverse order of the previous problem.

$$\begin{aligned} F_x(R_{60}(\bar{v})) &= (F_x \circ R_{60})(\bar{v}) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

This is the standard matrix for $(F_x \circ R_{60})(\bar{v})$.

3.6 Inverses of linear transformations

Definition 20. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, then T^{-1} is the inverse linear transformation, if

$$\begin{aligned} T^{-1}(T(\bar{v})) &= \bar{v} \\ T(T^{-1}(\bar{v})) &= \bar{v} \end{aligned}$$

Let A be the standard matrix of T, then T has an inverse T^{-1} if and only if A has an inverse. Furthermore, the standard matrix of the inverse T^{-1} is A^{-1}.

$$\begin{aligned} T^{-1}(T(\bar{v})) &= T^{-1}(A\bar{v}) \\ &= A^{-1}(A\bar{v}) \\ &= (A^{-1}A)\bar{v} \\ &= I\bar{v} \\ &= \bar{v} \end{aligned}$$

Example 36. Let $R_{60} : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by 60 degrees. What is the inverse R_{60}^{-1}?

See notes for drawing

We are looking for something that rotates by a negative 60 degrees.

$$\begin{aligned} R_{60}^{-1} &= R_{-60} \\ &= \begin{bmatrix} \cos(-60) & -\sin(-60) \\ \sin(-60) & \cos(-60) \end{bmatrix} \\ &= \begin{bmatrix} \cos(60) & \sin(60) \\ -\sin(60) & \cos(60) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Lets check

$$R_{60}^{-1}(R_{60}(\bar{v})) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 37. Find the inverse of the reflection F_x. We are looking for F_x^{-1}. Since the reflection happening a second time returns the vector to its original position, it is its own inverse. The standard matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

You can check this by multiplying it by itself, and it returns the identity matrix I.

Example 38. Does projection onto the line ℓ (through the origin) have an inverse (in the plane)? $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$.

See notes for drawing.

The standard matrix of P_ℓ is

$$P_\ell = \begin{bmatrix} d_1^2 & d_1 d_2 \\ d_1 d_2 & d_1^2 \end{bmatrix}$$

Where $\hat{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ is a unit direction vector for ℓ.

Since there is an infinite number of vectors that will project to the new vector on ℓ, so there is no inverse. Also, since the standard matrix P_ℓ is invertible, P_ℓ^{-1} does not exist.

Apr 09

Example 39. *See problem 26 in 3.6 of Poole*

If the angle between ℓ and the positive x-axis is θ, show that the matrix of F_ℓ is

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

See notes for drawing

We can rotate the entire plane so it is then a reflection about the x-axis.

$$\begin{aligned}
R_\theta(F_x(R_\theta^{-1}(\bar{v}))) &= R_\theta(F_x(R_{-\theta}(\bar{v}))) \\
&= (R_\theta \circ F_x \circ R_{-\theta})(\bar{v}) \\
&= F_\ell(\bar{v}) \\
&= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\text{standard matrix of } F_\ell} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ \sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}
\end{aligned}$$

Aside 1.

$$\begin{aligned}
\cos 2\theta &= \cos(\theta + \theta) \\
&= \cos^2\theta - \sin^2\theta
\end{aligned}$$

4 Eigenvalues and Eigenvectors

Definition 21. Let A be a $n \times n$ matrix. A scalar λ is an eigenvalue of the matrix A if there is a non-zero vector $\bar{v}$ such that

$$A\bar{v} = \lambda\bar{v}$$

where $\bar{v}$ is an eigenvector associated with λ.

Eigenvector can be abbreviated e-vector, and eigenvalue can be abbreviated e-value.

> **Note 11.** If λ is real, then the new vector will be parallel to the original vector. It is possible that λ is complex.

Example 40. Show that $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector of the matrix $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$ and find its eigenvalue.

$$\begin{aligned} A\bar{v} &= \lambda\bar{v} \\ \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} &= \begin{bmatrix} 8 \\ -12 \end{bmatrix} \\ &= 4\begin{bmatrix} 2 \\ -3 \end{bmatrix} \end{aligned}$$

So $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an e-vector with an e-value of $\lambda = 4$.

Example 41. Show that $\lambda_1 = -2$ and $\lambda_2 = 5$ are e-values of the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and find associated e-vectors.

We'll start with $\lambda_1 = -2$:

$$\begin{aligned} A\bar{v}_1 &= -2\bar{v}_1 \\ A\bar{v}_1 + 2\bar{v}_1 &= \bar{0} \\ A\bar{v}_1 + 2I\bar{v}_1 &= \bar{0} \\ (A + 2I)\bar{v}_1 &= \bar{0} \end{aligned}$$

So $\bar{v}_1$ is in the null space of $A + 2I$.

Aside 2.

$$\begin{aligned} 2I &= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A + 2I &= A - \lambda I \\ &= \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix} \end{aligned}$$

We are looking for $\bar{v}$ that is in the null space.

$$\begin{aligned} \begin{bmatrix} A + 2I & \bar{0} \end{bmatrix} &= \begin{bmatrix} 4 & 3 & 0 \\ 4 & 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ 4x + 3y &= 0 \end{aligned}$$

Let $\bar{v_1} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$, then $\bar{v_1} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ is an e-vector for $\lambda_1 = -2$. We can check this by

$$\begin{aligned} A\bar{v_1} &= -2\bar{v_1} \\ \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} &= -2\begin{bmatrix} 3 \\ -4 \end{bmatrix} \\ &= \begin{bmatrix} -6 \\ 8 \end{bmatrix} \\ &= -2\begin{bmatrix} 3 \\ -4 \end{bmatrix} \end{aligned}$$

Apr 14

Example 42. *See handout 16*

Apr 16

Example 43. *See handout 16 example 2*

5 Determinants

See handout 18

5.1 Cofactor expansion

Example 44. *See handout 18 example at end on cofactors* Apr 19

5.2 Invertibility

Definition 22. If a matrix A is full rank and square ($n \times n$), then it will row reduce to the identity matrix $I_{n\times n}$. Therefore,

- The matrix is invertible.

$$\left[A \mid I\right] \to \left[I \mid A^{-1}\right]$$

- The determinant is non-zero.

Less than full rank $n \times n$ matrices row reduce to a row of zeros at the bottom of the matrix. Therefore, Apr 21

- It will have a zero determinant.
- It will not be invertible.

Theorem 6. The $n \times n$ matrix A is invertible if and only if $det(A) \neq 0$.

See more theorems in handout 18

5.3 Cramer's rule

Definition 23. *See handout 18*

Let A be an invertible $n \times n$ matrix, and let $\bar{b}$ be any vector in $\mathbb{R}^n$. Then the unique solution $\bar{x}$ of the system $A\bar{x} = \bar{b}$ is given by

$$x_i = \frac{det(A_i(b))}{detA}$$

for $i = 1, \ldots, n$.

Note that $A_i(b)$ is created by replacing the ith column of A with the vector $\bar{b}$.

Example 45. *See handout 18, example on Cramer's rule*

5.4 Determinants and Eigenvalues

See handout 19 Apr 23

To find the eigenvalues and eigenvectors:

1. Find λ such that $det(A - \lambda I) = 0$.
2. Substitute into the equation

$$[A - \lambda I]\bar{v} = \bar{0}$$

and solve for $\bar{v}$.

Example 46. *See handout 19 example 1*

Example 47. *See handout 19 example 2a/b*

Apr 26

Example 48. *See handout 19 example 3*

Example 49. *See handout 19 example 4*

5.5 Similarity and Diagonlization

Definition 24. For $n \times n$ matrices A and B, A is **similar** to B, written $A \sim B$, if an invertible $n \times n$ matrix P exists such that Apr 28

$$P^{-1}AP = B$$

Definition 25. An $n \times n$ matrix A is **diagonalizable** if there is a diagon matrix D that is similar to A.

Theorem 7. The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. (Deficient matrices need not apply!)

Example 50. *See handout 20 example 1*

Apr 30

Theorem 8. Let P be the matrix whose columns are independent eigenvectors of matrix A. Then the entries of diagonal marix $D = P^{-1}AP$ are the eigenvalues of A.

Proof:

Let P be an invertible matrix of eigenvectors of $A_{n\times n}$. Let $\bar{P}_j$ be the jth column of vector P.

$$P = \begin{bmatrix} \bar{P}_1 & \cdots & \bar{P}_n \end{bmatrix}$$

Then

$$\begin{aligned} P^{-1}P &= P^{-1}\begin{bmatrix} \bar{P}_1 & \cdots & \bar{P}_n \end{bmatrix} \\ &= \begin{bmatrix} P^{-1}\bar{P}_1 & \cdots & P^{-1}\bar{P}_n \end{bmatrix} \\ &= \begin{bmatrix} \bar{e_1} & \cdots & \bar{e_n} \end{bmatrix} \\ &= I_{n\times n} \end{aligned}$$

Now,

$$\begin{aligned} P^{-1}AP &= P^{-1}A\begin{bmatrix} \bar{P}_1 & \cdots & \bar{P}_n \end{bmatrix} \\ &= P^{-1}\begin{bmatrix} A\bar{P}_1 & \cdots & A\bar{P}_n \end{bmatrix} \\ &= P^{-1}\begin{bmatrix} \lambda_1\bar{P}_1 & \cdots & \lambda_n\bar{P}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 P^{-1}\bar{P}_1 & \cdots & \lambda_n P^{-1}\bar{P}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1\bar{e_1} & \cdots & \lambda_n\bar{e_n} \end{bmatrix} \\ &= \lambda I_{n\times n} \end{aligned}$$

Where $\lambda I_{n\times n}$ is the corresponding eigenvalues along the diagonal of I. So, $A \sim D$ where the diagonal entries of D are the corresponding eigenvalues.

Example 51. *See handout 20 example 3*

Example 52. *See handout 20 example 4*

May 03

6 Distance and approximation

6.1 Least squares approximation

See handout 21

Recall our $A\bar{x} = \bar{b}$ problem, where A is a $m \times m$ matrix, and $\bar{x}$ is what we're solving for.

Recognizing that $A\bar{x} = \bar{b}$ has no solution for most *overdetermined* systems, we transform the problem into a related (but different) problem,

$$A^T A\tilde{x} = A^T\bar{b}$$

Note 12. Overdetermined systems are when we have more equations than variables. It is also certain

> that we don't have a solution because we have too many constraints on the variables.

We are only considering the case where A is full rank, $rank(A) < min\{m, n\}$ for skinny matrices, $m > n$, $rank(A) \leq n$, where A is full rank if $rank(A) = n$, if and only if the columns of A form a linearly independent set.

$$rank(A^T A) = rank(AA^T) = rank(A) = n$$

$\tilde{x}$ is called the *least squares approximation* for $A\bar{x} = \bar{b}$.

Date Index