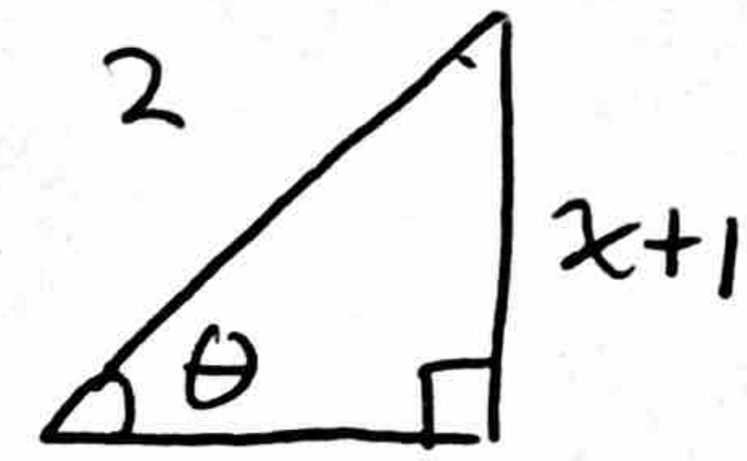
ADJ → $\sqrt{4-(x+1)^2}$

Math 31 | Exam 1 Techniques of Integration Fall 2018

Calculate the following integrals: (20 points EACH)

1. $\int \sqrt{3-2x-x^2}\, dx$ [7.5.33]



$\sin\theta = \dfrac{x+1}{2}$

$2\sin\theta = x+1$

$2\cos\theta\, d\theta = dx$

COMPLETE SQUARE

$$= \int \sqrt{4-1-2x-x^2}\, dx$$

$$= \int \sqrt{4-(x+1)^2}\, dx$$

$$= \int \sqrt{4}\sqrt{1-\sin^2\theta}\,(2)\cos\theta\, d\theta$$

TRIG SUB

$$= 4\int \cos^2\theta\, d\theta$$

$$= \frac{4}{2}\int (\cos 2\theta + 1)\, d\theta$$

TRIG INT

$$= 2\left[\frac{1}{2}\sin 2\theta + \theta\right] + C$$

$$= 2\sin\theta\cos\theta + 2\theta + C$$

RE-SUB

$$= 2\left[\frac{x+1}{2}\right]\left[\frac{\sqrt{4-(x+1)^2}}{2}\right] + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= \boxed{\frac{1}{2}(x+1)\sqrt{4-(x+1)^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + C}$$

2. $\int_0^2 \ln(4+x^2)\,dx = I$ [HCB.14.45]

$u = \ln(4+x^2) \qquad dv = dx$

$du = \frac{1}{4+x^2}\cdot 2x \qquad v = x$

$\boxed{\text{IBP}}$

$uv - \int v\,du$

$$= x\ln(4+x^2)\Big|_0^2 - \int_0^2 \frac{2x^2}{x^2+4}\,dx$$

$\boxed{\text{PRF}}$

$$= x\ln(4+x^2)\Big|_0^2 - \int_0^2 \frac{2x^2+8-8}{x^2+4}\,dx$$

$$= x\ln(4+x^2)\Big|_0^2 - \int_0^2 2 - \frac{8}{x^2+4}\,dx$$

$$= x\ln(4+x^2)\Big|_0^2 - 4 + 8\int \frac{1}{4\left(\left(\frac{x}{2}\right)^2+1\right)}\,dx$$

$\boxed{\tan^{-1} \text{ or TRIG INT}}$

$$= x\ln(4+x^2)\Big|_0^2 - 4 + 4\tan^{-1}\left(\frac{x}{2}\right)\Big|_0^2$$

$$= 2\ln(8) - 4 + 4\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= 2\ln(8) - 4 + 4\left(\frac{\pi}{4}\right)$$

$\boxed{\text{SIMP}}$

$$= \boxed{\ln 64 - 4 + \pi}$$

3. $\int \frac{1}{x + x\sqrt{x}}\,dx$ [7.5.55]

$\boxed{\text{RAT SUB}}$

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$= \int \frac{1}{x(1+\sqrt{x})}\,dx$$

$$= \int \frac{2u}{u^2(1+u)}\,du$$

$$= \int \frac{1}{u(1+u)}\,du$$

$1 = A(1+u) + Bu$

$1 = -B$

$1 = A$

$\boxed{\text{PFD}}$

$$= 2\int \frac{A}{u} + \frac{B}{1+u}\,du$$

$$= 2\int \frac{1}{u} + \frac{-1}{1+u}\,du$$

$\boxed{\text{INT}}$

$$= 2\ln|u| - 2\ln|1+u| + C$$

$$= 2\ln\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right) + C$$

$\boxed{\text{RE-SUB}}$

4. $\int \frac{6x+2}{x^4-1}\,dx$ [HCB.14.34]

$$= \int \frac{6x+2}{(x-1)(x+1)(x^2+1)}\,dx$$

$$= \int \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}\,dx = I$$

$$6x+2 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x^2-1)$$
$$= A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + Cx^3 + Dx^2 - Cx - D$$

$\therefore$ WE HAVE

$0 = A + B + C$

$0 = A - B + D$

$6 = A + B - C$

$2 = A - B - D$

IF $x = 1$ THEN

$6(1) + 2 = 4A$

$\therefore A = 2$

IF $x = -1$ THEN

$6(-1) + 2 = B(-2)(2)$

$-4 = -4B$

$\therefore B = 1$

$\therefore A + B + C = 0$

$2 + 1 + C = 0$

$\therefore C = -3$

$\therefore A - B + D = 0$

$2 - 1 + D = 0$

$\therefore D = -1$

$$I = \int \frac{2}{x-1} + \frac{1}{x+1} - \frac{3x+1}{x^2+1}\, dx$$

$$= 2\int \frac{1}{x-1}\, dx + \int \frac{1}{x+1}\, dx$$

$$- \frac{3}{2}\int \frac{2x}{x^2+1}\, dx - \int \frac{1}{x^2+1}\, dx$$

$$= 2\ln|x-1| + \ln|x+1| - \frac{3}{2}\ln(x^2+1) - \tan^{-1}(x) + C$$

$$= \ln \frac{(x-1)^2|x+1|}{\sqrt{(x^2+1)^3}} - \tan^{-1}(x) + C$$

5. $\int_0^{\pi} x\sin^2 x\cos x\,dx = I$ [7.5.79]

$u = x \qquad dv = \sin^2 x\cos x\,dx$

$du = dx \qquad v = \int \sin^2 x\cos x\,dx$

$w = \sin x\,dx$
$dw = \cos x\,dx$

$= \int w^2\,dx$

CHOOSE $C=0$.

$= \frac{1}{3}w^3 + C = \frac{1}{3}\sin^3 x$

$$I = uv - \int v\,du$$

$$= \frac{1}{3}x\sin^3 x\Big|_0^{\pi} - \frac{1}{3}\int_0^{\pi}\sin^3 x\,dx$$

$$= \frac{1}{3}\pi(\sin\pi)^3 - \frac{1}{3}\int_0^{\pi}\sin x\sin^2 x\,dx$$

$$= \frac{\pi}{3}(0)^3 - \frac{1}{3}\int_0^{\pi}\sin x(1-\cos^2 x)\,dx$$

$z = \cos x$
$dz = -\sin x\,dx$

$$= -\frac{1}{3}\int_0^{\pi} -(1-z^2)\,dz$$

$z(0) = 1$
$z(\pi) = -1$

$$= -\frac{1}{3}\left(z - \frac{1}{3}z^3\right)\Big|_{x=0}^{x=\pi}$$

$$= -\frac{1}{3}\left[(1-(-1)) - \frac{1}{3}(1^3 - (-1)^3)\right]$$

$$= -\frac{1}{3}\left[2 - \frac{2}{3}\right]$$

$$= -\frac{1}{3}\cdot\frac{4}{3} = \boxed{-\frac{4}{9}}$$