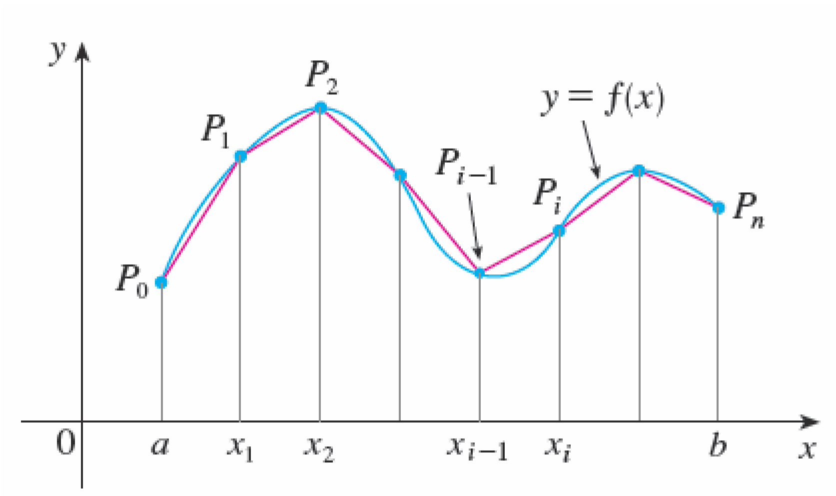
§8.1 | Arc Length

A curve is to be **smooth** if the curve is continuously differentiable. That is, it's derivative is a continuous function. If a function f defines a smooth curve in the plane then the Length, L, of the curve from a to b is define by,

$$L = \int_a^b \sqrt{1+\left(f'(x)\right)^2}\,dx\,.$$

The build looks like this:



$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

Note: The curve need only be piece-wise smooth for the length to exist.

Example 1: Find the length of the arc of a parabola $x = y^2$ from (0,0) to (1,1)

Example 2: Find the length of the arc $y = \ln(x+\sqrt{x^2-1})$ on $\left[1,\sqrt{2}\right]$ (switching variables)

Example 3: How long is a sine curve on one cycle? (cautionary tale)

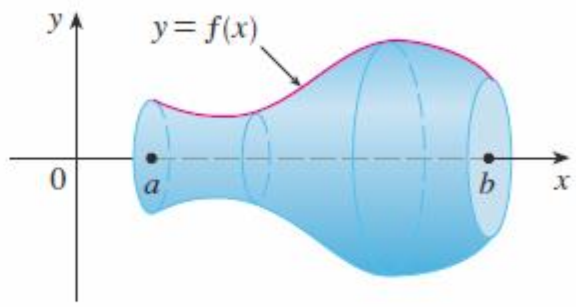
Example 4: Find the length of the arc $y = \sqrt{x-x^2}+\sin^{-1}(\sqrt{x})$ (have courage!)

Example 5:

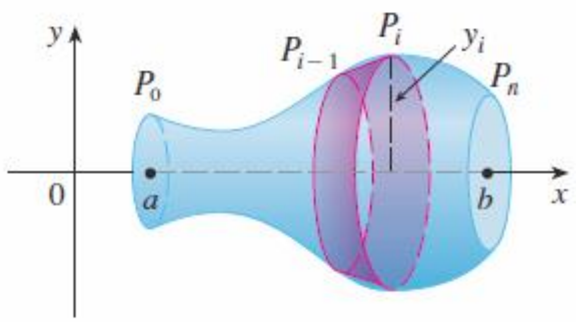
Example 6: Find the Arc Length Function for $f(x) = x^2 - \frac{1}{8}\ln x$. (Important fro Math 32!)

§8.2 | Area of a Surface of Revolution

The surface area of revolution is very similar in its derivation to the arc length of a function.



(a) Surface of revolution



(b) Approximating band

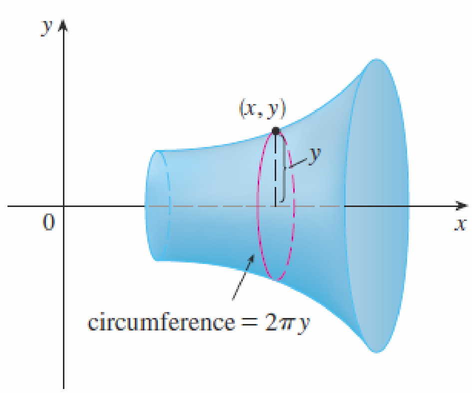
$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

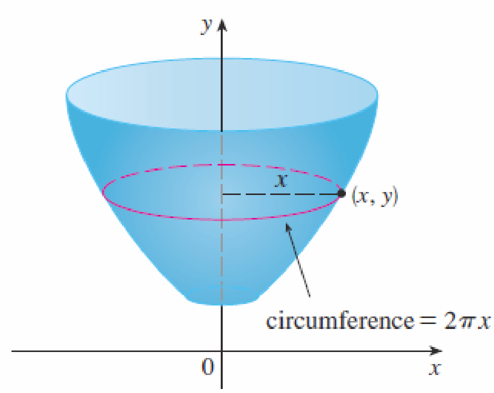
$$S = \int 2\pi y\, ds$$

$$S = \int 2\pi x\, ds$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$



(a) Rotation about x-axis: $S = \int 2\pi y\, ds$



(b) Rotation about y-axis: $S = \int 2\pi x\, ds$

Example 1: Find the surface area of the curve $y = \sqrt{4 - x^2} - 1$ between −1 and 1.

Example 2: Find the surface area of a sine curve on one cycle?

Example 3: Find the surface area of revolution by rotating the parabola $y = x^2$ from the points (1,1) to (2, 4) about the y-axis.

Example 4: Find the surface area of revolution by rotating the curve $y = e^x$ with $0 < x < 1$ around the x-axis. (What about the y-axis?)

If you are interested in how are length and surface area gets built-up I will provide a handout in class.

§8.3 | Applications to Physics and Engineering

Hydrostatic pressure is the **pressure** exerted by a fluid at equilibrium at a given point within the fluid, due to the force of gravity. Generally, we define pressure to be force per unit of area.

- **Hydrostatic pressure** increases in proportion to depth measured from the surface because of the increasing weight of fluid exerting downward force from above.
- **Hydrostatic pressure** is the same in all directions in the fluid. In a sense, gravity is pulling down on a body of fluid and since the fluids we are considering are in a container, the force get's exerted in all directions.

Derivation

We start with force.

$$F = ma$$
$$F = \rho V a$$
$$F = \rho A d a$$
$$F = \rho A d g$$

Pressure (P) is just force (F) divided by area (A)

$$P = \frac{F}{A}$$

$$P = \frac{\rho A d g}{A}$$
$$P = \rho d g$$

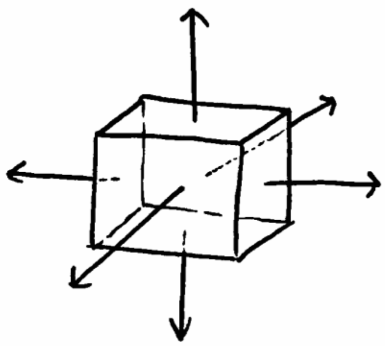
In the language of Calculus we'd have the hydrostatic force exerted on a surface by a fluid body with a depth of a, a density of ρ, and tank width w(y) as

$$F = \int_0^a \rho g(a-y)w(y)\,dy$$

with g representing the acceleration due to gravity. The Hydrostatic Pressure (P) on the surface would be

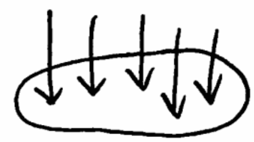
$$P = \frac{F}{A} = \frac{\int_0^a \rho g(a-y)w(y)\,dy}{\int_0^a (a-y)w(y)\,dy}$$

HYDROSTATIC PRESSURE



WATER PUSHES IN ALL DIRECTIONS,

PRESSURE (P)



IF FORCE (F) PER AREA (A)

$$P = \frac{F}{A}$$

$$= \frac{(ma)}{A}$$

$$= \frac{\rho V g}{A}$$

$$= \rho \frac{dA}{A} g$$

$$= \rho d g$$

WHERE ρ IS DENSITY, d IS THE DEPTH & g IS ACCELERATION DUE TO GRAVITY.

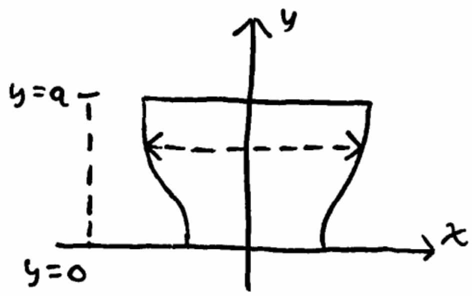
$$F = PA$$

$$F_i = P_i A_i$$

$$\approx [\rho g (a - y_i)][w(y_i) \Delta y]$$

$$F = \sum_{i=1}^{\infty} F_i$$

$$= \sum_{i=1}^{\infty} P_i A_i$$



$$\boxed{F = \int_0^a \rho g (a-y) w(y) \, dy}$$

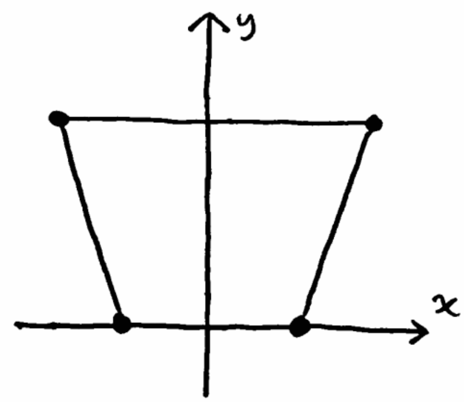
W/ AXIS @ BOTTOM OF TANK, TANK OF DEPTH a AND $w(y)$ AS THE WIDTH OF THE TANK.

Example 1

PRESSURE ON A DAM.

A LARGE VERTICAL DAM IN THE SHAPE OF A SYMMETRIC TRAPEZOID HAS A HEIGHT OF 30 M, WIDTH OF 20 M @ IT'S BASE & A WIDTH OF 40 M @ IT'S TOP. WHAT IS THE TOTAL FORCE ON THE FACE OF THE DAM, WHEN FULL?

$$F = \int_0^a \rho g (a-y) w(y) \, dy$$



$a =$

$a - y =$

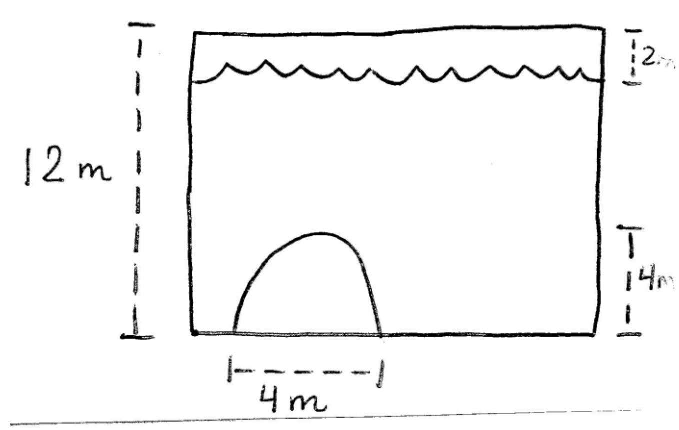
$w(y) = 2(\qquad)$

Example 2

A diving pool that is 4m deep & full of water has a viewing window on one of its walls. Find the force on a window that is square, 0.5 m on one side, with the lower edge of the window 1 m from the bottom of the pool.

Example 3

A vertical dam has a parabolic gate as shown in the diagram. Find the hydrostatic force and pressure on the gate.



Example 4

A semicircular vertical plate of diameter 12 m is partially submerged in water at a depth of 4 m. (let’s draw a picture). Find the hydrostatic force on the plate.

Note: We will not lecture on moments and centers of mass in §8.3, but there will be a take-home assignment. We will also not lecture on §8.4 (Applications to Economics & Biology) & §8.5 (Probability) but I will have a take-home assignment for both.