Needed Algebra, Trigonometry and Calculus Skills

Algebra

Trigonometry

Double Angle Identities (*Half-Angle Identities*)

$$\sin 2\theta =$$
$$\cos 2\theta =$$
$$\cos 2\theta =$$
$$\cos 2\theta =$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$=$$
$$=$$

Calculus

Review of Integration

Question: How many types of integrals are there?

Question: What techniques do we have to address each type?

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

§5.5 The Substitution Rule

Warm-up: Calculate $\frac{d}{dx}\sin(5x^2)$ and use your answer to calculate $\int 10x\sin(5x^2)dx$ and $\int 3x\sin(5x^2)dx$.

Substitution is one of the most useful techniques for solving integrals. Put simply substitution is to the chain rule as integration is to differentiation.

> **4 THE SUBSTITUTION RULE** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then
>
> $$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

When using substitution we should remember a few things:

- We may need to use the technique multiple times.
- There is often more then on substitution that could be used.
- Think strategically.
- When dealing with a definite integral, remember to change the limits of integration *or* re-substitute.

Substitution for Indefinite Integral

Example 1: Calculate $\int (x^2+1)(x^3+3x)^4\,dx$.

Example 2: Calculate $\int \cot(x)\,dx$.

Example 3: Calculate $\int e^{3x}dx$.

Example 4: Calculate $\int \sin 3x\cos^2 3x\,dx$.

Example 5: Calculate $\int \frac{x}{\sqrt{1-2x^2}}\,dx$.

Substitution for Definite Integral

6 THE SUBSTITUTION RULE FOR DEFINITE INTEGRALS If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 6: Calculate $\displaystyle\int_0^{\sqrt{3}} \frac{\tan^{-1}(x)}{x^2+1}\,dx$.

Example 7: Calculate $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$.

Example 8: Calculate $\displaystyle\int_0^{\frac{\sqrt{\pi}}{2}} x\sec(x^2)\tan(x^2)\,dx$.

Example 9: Calculate $\displaystyle\int_0^{\sqrt{3}} \frac{\tan^{-1}(x)}{x^2+1}\,dx$.

Advanced Substitution

Example 10: Calculate $\displaystyle\int_0^2 \frac{x^3}{(x+1)^4}\,dx$.

Example 11: Calculate $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$.

DIFFERENTIATION IS A SKILL, INTEGRATION IS AN ART!

§7.1 Integration by Parts

Integration by parts is our first new technique for integration beyond knowing our derivatives, guessing & checking, geometry and the **substitution method**.

Recall the product rule for derivatives,

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x).$$

If we manipulate this equation and use FTC, we have

$$\begin{aligned}
\frac{d}{dx}\left[f(x)g(x)\right] &= f(x)g'(x) + g(x)f'(x) \\
f(x)g'(x) + g(x)f'(x) &= \frac{d}{dx}\left[f(x)g(x)\right] \\
f(x)g'(x) &= \frac{d}{dx}\left[f(x)g(x)\right] - g(x)f'(x) \\
\int f(x)g'(x) &= \int\left(\frac{d}{dx}\left[f(x)g(x)\right] - g(x)f'(x)\right)dx \\
&= \int \frac{d}{dx}\left[f(x)g(x)\right]dx - \int g(x)f'(x)\,dx \\
&= f(x)g(x) - \int g(x)f'(x)\,dx
\end{aligned}$$

This is **integration by parts** and is typically remembered in the following form,

$$\int u\,dv = uv - \int v\,du$$

We must choose a factor of the integrand to *integrate* (dv) and a factor to *differentiate* (u). Like substitution, we have many options, not all of which will work out well.

Caution: Do not forget your limits of integration for definite integrals when using IBP.

A useful acronym for trying to figure out what to choose for your u & v is LIATE.

	Function	Examples
L	Logarithmic Functions	
I	Inverse Trig. Functions	
A	Algebraic Functions	
T	Trig. Functions	
E	Exponential Functions	

This is sometimes referred to as the tabular method. One word of caution, this acronym does not always work.

Examples

Evaluate the following integrals

1. $\int x \sin x dx$
2. $\int \ln(x) dx$
3. $\int t^2 e^t dt$
4. $\int_0^1 t^2 e^t dt$
5. $\int_0^1 \arctan(x) dx$
6. $\int_0^2 y \sinh y \, dy$
7. $\int e^x \sin x \, dx$
8. $\int x^2 \ln x \, dx$

Integration by parts (IBP) is another tool we use to integrate along with knowing our derivatives, algebraic manipulations, guessing and checking, symmetry, geometry and substitution. Integration by parts requires us to identify a factor of the integrand to differentiate, identify a factor to integrate and remember the form of IBP. Not every choice of u & v will work and often we have to use IBP multiple times or in clever ways. IBP often works to integrate those integrals that look easy but in fact are not.

§7.2 Trigonometric Integrals

Integrals of functions comprised strictly of trigonometric functions utilize several key trigonometric identities. We are trying to use these identities and the other techniques of integration to solve trigonometric integrals. We need to be strategic!

Double Angle Identities (*Half-Angle Identities*)

$$\sin 2\theta =$$
$$\cos 2\theta =$$
$$\cos 2\theta =$$
$$\cos 2\theta =$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$=$$
$$=$$

I have never memorized these identities. I typically have to derive them every time. I am not good at memorizing. That said, you might want to memorize them.

Strategy for Evaluating Integrals with Trigonometric Functions

Our core strategy will be to use trigonometric identities and substitution to turn an integral with trigonometric functions into an integral with just polynomials. We will focus much of our class time on sine & cosine for two reasons. First, the same “game” plays out with tangent & secant as well as cosecant and cotangent. (Why?). Second, very often you can turn all trig functions into just sines and cosines. Word of caution: changing every trigonometric functions in sines and cosines is not always the optimal strategy.

Here is a nice chart from our textbook that synthesizes the general approach to integrals with sines and cosines.

STRATEGY FOR EVALUATING $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x\, (\cos^2 x)^k \cos x\, dx$$

$$= \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

My advice is not to memorize this chart, but rather understand how it works and it's general *spirit*. Also, similar games work with sec/tan and csc/cot.

Examples
Evaluate the following integrals.

1. $\int \cos^3 x\, dx$
2. $\int \sin^5(x)\cos^2(x)\, dx$
3. $\int_0^{\pi} \sin^2 x\, dx$
4. $\int \sin^4(x)\, dx$
5. $\int \tan^5 y \sec^7 y\, dy$
6. $\int \sec^3 x\, dx$

Easy (Math 30 Level)
Sine & Cosine
Secant & Tangent
Cosecant & Cotangent
Mixed Function

Summary
We try to use trigonometric identities whenever possible and strategically to evaluate trigonometric integrals. Any trigonometric identity is fair game. Sometimes changing everything into just sines and cosines is a place to start if we are stuck. Substitution is the name of the game, but even IBP is fair game.

§7.3 Trigonometric Substitution

Trigonometric substitution is a special kind of substitution that requires us to be able to identify factors of the integrand of a certain form. Essentially, trigonometric substitution is an *inverse substitution*.

Example 1: $\int \sqrt{9+x^2}\, dx$

There are three main types of trigonometric substitution.

Expression	Substitution	Identity used
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$1-\sec^2\theta = \tan^2\theta$

A few things to remember when using an inverse trig. substitution:

- You have identify which form you are dealing with and this is largely based off of the identity you are trying to use.
- Since this is an inverse substitution, you have to change you “dx” to some function of θ times $d\theta$.
- Keep an eye on your limits of integration when dealing with a definite integral.
- Resubstitute if calculating an indefinite integral. This part is typically not trivial.
- Why didn’t cotangent and cosecant come up?

Examples

Evaluate the following integrals

1. $\int \frac{\sqrt{9-x^2}}{x^2}dx$
2. $\int \frac{1}{x^2\sqrt{x^2+4}}dx$
3. $\int_0^{\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}dx$
4. $\int \frac{x}{\sqrt{3-2x-x^2}}dx$
5. Find the area enclosed by an ellipse in the first quadrating of the following form:

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

§7.4 Integration of Rational Functions by Partial Fractions

Partial fraction decomposition (PFD) at its essence is to "un-add" fractions.

Example: Integrate $\int \frac{2}{(x-2)(x+3)}\,dx$.

Before we *really* get into the meat of PFD, there are a few cases of integrating ration functions that don't need PFD.

Example: Integrate $\int \frac{1}{x^2+4x+13}\,dx$. [HCB.14.17]

Example: Integrate $\int \frac{2x+1}{3x^2+4x+2}\,dx$. [HCB.14.19]

Example: Integrate $\int \frac{x^3+x}{x-1}\,dx$. [7.4.E1]

Partial Fraction Decomposition

PFD breaks down into four cases.

<u>Case #1</u>: Distinct Linear Factors. (First example above)
<u>Case #2</u>: Repeated Linear Factors.

Example: Integrate $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$. [7.4.E4]

<u>Case #3</u>: Distinct Irreducible Quadratic Factors.

Example: Integrate $\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}dx$. [7.4.33]

<u>Case #4</u>: Repeated Irreducible Quadratic Factors.

Example: Integrate $\int \frac{x}{x^5 - x^4 + 2x^3 - 2x^2 + x - 1}dx$. [HCB.14.22]

A few things to remember when using PFD

- PFD works on rational functions.
- We must be dealing with a proper rational function. (*degree of denominator must be greater than numerator*)
- We will be solving systems of equations.
  - Back Substitution
  - Matrix (Don't be afraid!)
  - "Cover-up Method
- Knowing the following integrals is very useful:

$$\int \frac{1}{x^2 + a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$\int \frac{1}{x}dx = \ln|x| + c$$

Examples

Use PFD to solve the following integrals

1. Integrate $\int \frac{5x+1}{(2x+1)(x-1)}\,dx$
2. Integrate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

§7.5 Strategy for Integration

<u>Math 30 Techniques for Integration</u>

- Simplify expressions. (Integration prefers terms!) (D/I)
- Even/Odd Symmetry (D)
- Geometry (D)
- Substitution (D/I)
    - Rationalizing Substitution
- The limit/sum definition of integral (D)

<u>Math 31 Techniques for Integration</u>

- Integration By Parts (IBP) (D/I)
- Trigonometric Identities (D/I)
- Trigonometric Substitution (D/I)
- Partial Fraction Decomposition (PFD) (D/I)

D = ______________________ I = ______________________

Historically in calculus education, there are a few more techniques: Integration tables, rational functions of sine & cosine, or the tabular method (add-on to IBP). We will not cover them but you are encouraged to check them out.

Each technique individually is not impossible to master. However, using more than one technique quickly and efficiently can very challenging and that is what you are being asked to do.