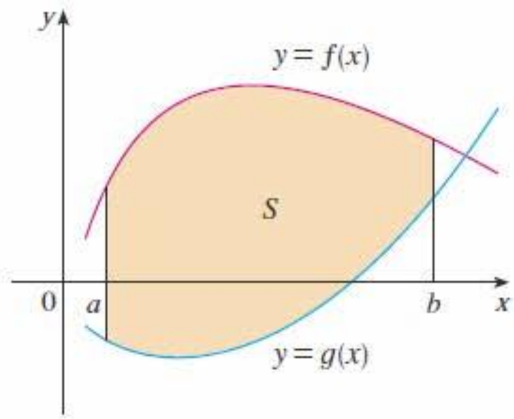
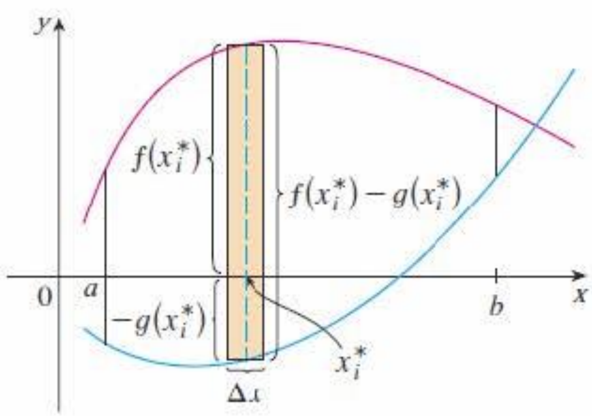
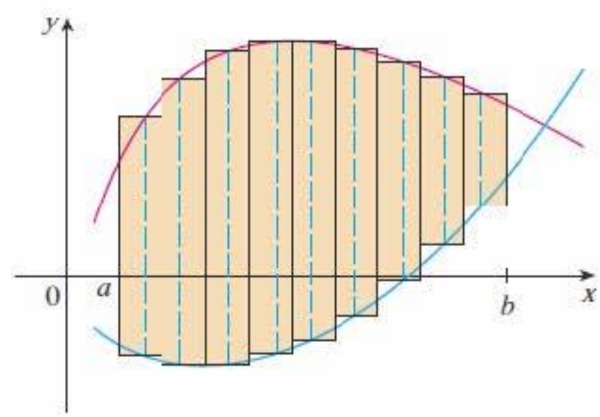
Math 31 | Lecture Notes | §6.1: Area Between Curves

A great way to characterize the definite integral is the oriented area bellow the graph of a curve. That is the area bellow the curve but above the x-axis. Area bellow the x-axis we consider to be “negative” even thou that seems very odd to us.

In the section, we will explore the area between two curves. This area we will always define to be *positive*.







$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [\, f(x_i^*) - \mathrm{t}(x_i^*)] \, \Delta x$$

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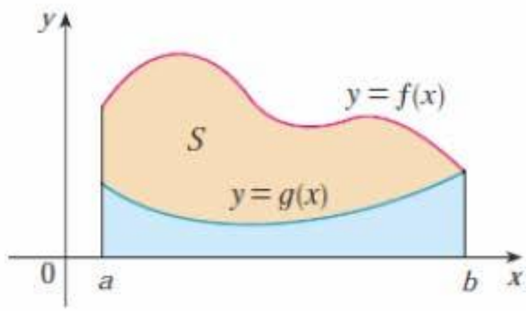
> **2** The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$, where f and t are continuous and $f(x) \geqslant t(x)$ for all x in $[a, b]$, is
>
> $$A = \int_a^b [f(x) - g(x)]\, dx$$



> **3** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ is
>
> $$A = \int_a^b |f(x) - g(x)|\, dx$$

These two definitions are essential the same, but which seems more appealing to you?

In determining the area between two curves, we must:

1. Determine where the curves meet (solve an equation)
2. Determine which curve is above the other (draw a picture)
3. Check to see if the corves cross each other multiple times

Example 1: Find the area between $y = x^2$ and $y = 2x - x^2$

Example 2: Find the area between $y = x - x^2$ and $y = x^3 - x$

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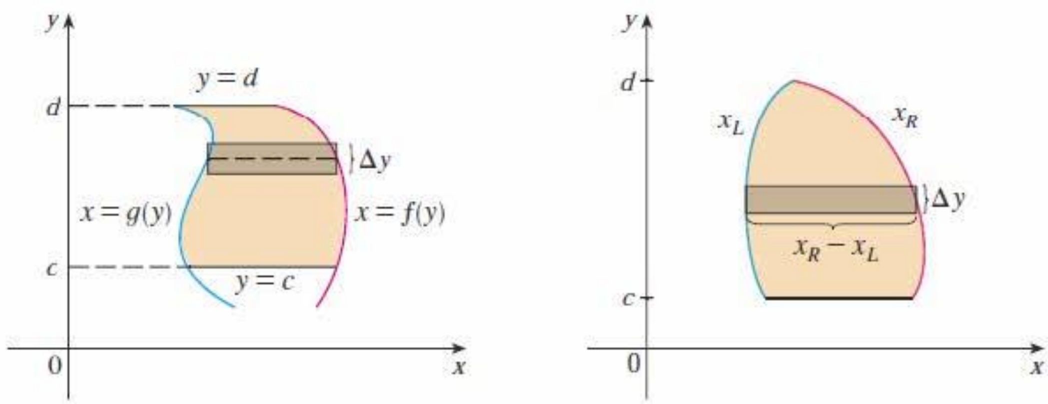
Regions as a Function of y

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$, where f and t are continuous and $f(y) \geqslant g(y)$ for $c \leqslant y \leqslant d$ (see Figure 11), then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$



Example 3: Find the area bounded by $y = x - 1$ and $y^2 - 2x = 6$.

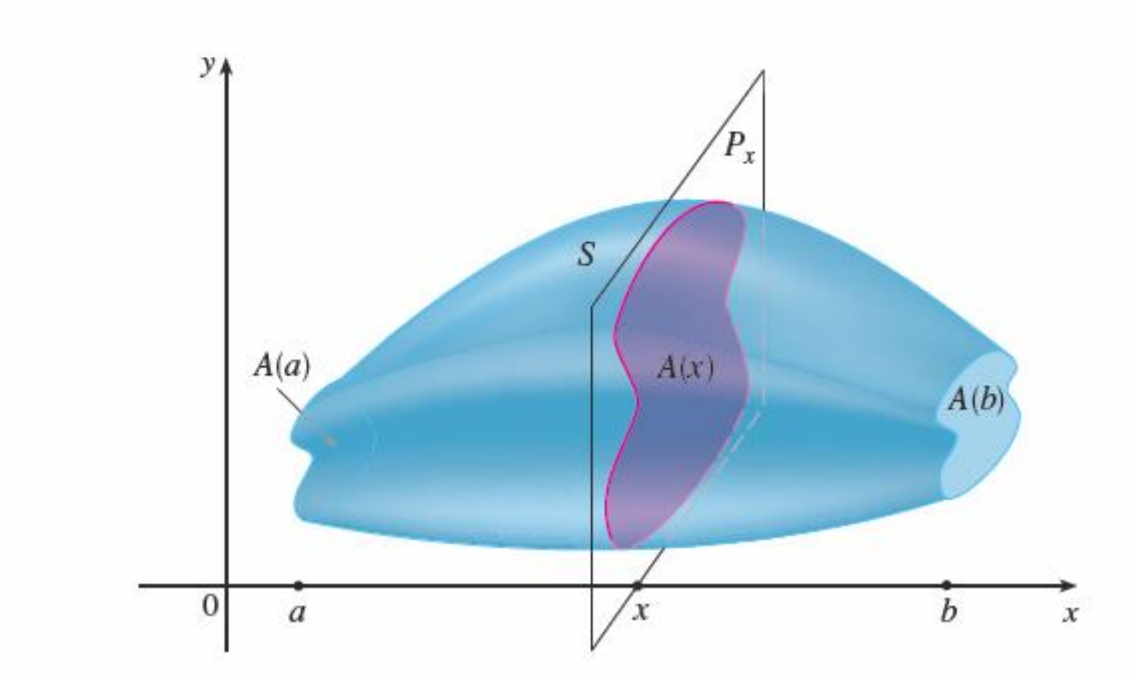
Example 4: Find the area bounded by $y = \cos x$ and $y = \sin 2x$ for $0 \le x \le \dfrac{\pi}{2}$.

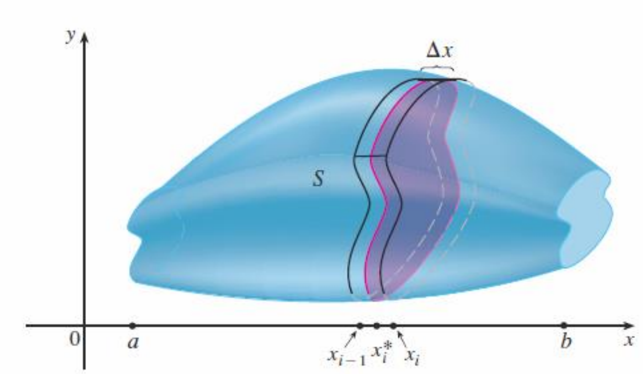
Example 5: Find the area bounded by $y = (x-1)^2$ and $x + y = 3$ in the first and

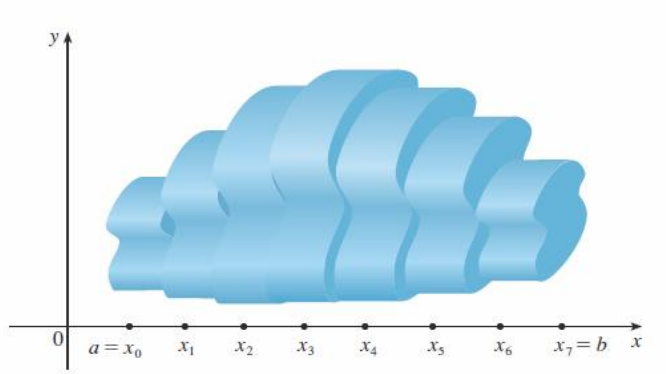
§6.2 | Volumes

Definition of Volume Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

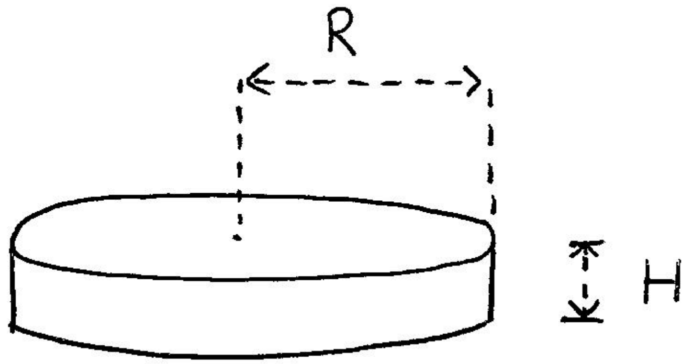
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$







Disk Method



WHAT IS THE VOLUME OF A DISK?

V =

EXAMPLE 1

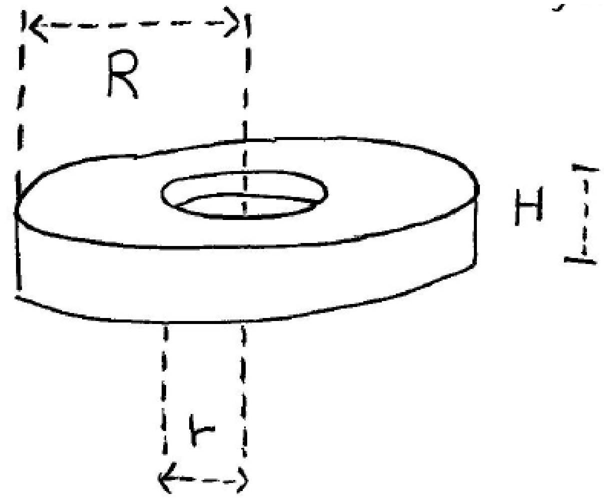
FIND THE VOLUME OF THE REGION BOUNDED BY $y = \sqrt{x}$ AND $x = 2$ REVOLVED AROUND THE x-AXIS. (DRAW A PICTURE)

EXAMPLE 2

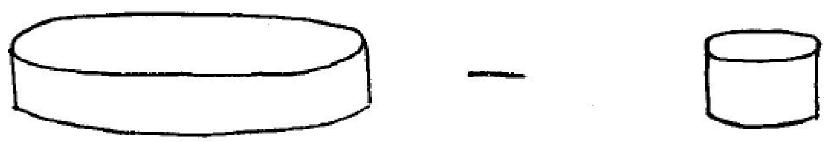
FIND THE VOLUME OF THE SAME BOUNDED REGION, BUT ROTATED AROUND THE y-AXIS. (DRAW A PICTURE)

Washer

WHAT IS THE VOLUME OF THIS WASHER?



$V =$



EXAMPLE 3

FIND THE VOLUME OF THE REGION BOUNDED BY $y=\sqrt{x}$ & $y=x$ REVOLVED AROUND THE X-AXIS.

EXAMPLE 4

FIND THE VOLUME OF THE REGION BOUNDED BY $y=\sqrt{x}$ & $y=x$ REVOLVED AROUND THE LINE $x=1$.

EXAMPLE 5

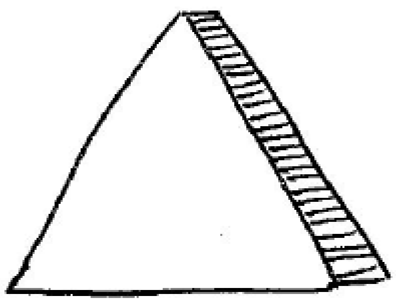
FIND THE VOLUME OF THE REGION BOUNDED BY $y=\sqrt{x}$ & $y=x$ REVOLVED AROUND THE LINE $y=-1$.

Watch out for which radius is the OUTER and which is the INNER!

Volume by Slicing

Volume by slicing is the more general technique associated with finding the volume of a revolution by using disks & washers.

EXAMPLE 6
FIND THE VOLUME OF A SOLID W/ A CIRCULAR BASE OF RADIUS 1. PARALLEL CROSS SECTIONS $\perp$ TO THE BASE ARE EQUILLATERAL TRIANGLES.



$V =$

Example 7: Find the volume of a solid with a circular base of radius 1 unit, with parallel cross-sections perpendicular the base as equilateral triangles.

Example 8: The base of an elliptical curve is $9x^2 + 4y^2 = 36$. Cross-sections perpendicular to the y-axis are isosceles right triangles with the hypotenuse at the base. Find the volume of this object.

Other Examples (ITP)

OTHER EXAMPLES

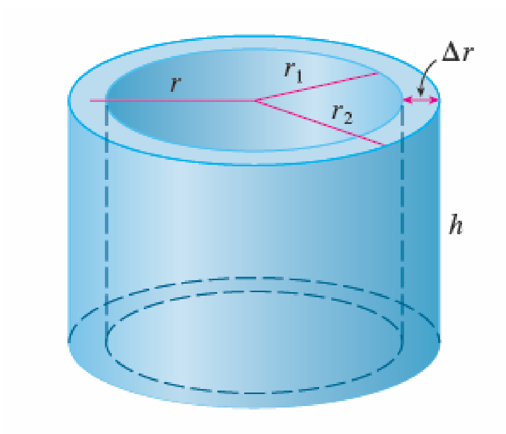
1. FIND THE VOLUME OF THE SOLID BOUND BY $y = \cos x$ & $y = 1 - \cos x$ IF $-\frac{\pi}{3} \le x \le \frac{5\pi}{3}$ REVOLVED AROUND THE LINE $y = -2$.

2. COMPUTE THE VOLUME OF THE SOLID WHOSE BASE IS A TRIANGLE W/ VERTICES AT $(0,0)$, $(2,0)$ & $(0,2)$ WHOSE CROSS SECTIONS ARE $\perp$ TO THE BASE & $\parallel$ TO Y AXIS ARE SEMI CIRCLES.

3. SAME BASE AS ABOVE, BUT CROSS SECTIONS ARE $\perp$ TO THE BASE & $\parallel$ TO THE X-AXIS ARE SQUARES.

§6.3 | Volumes Using Cylindrical Shells

The solids we wish to find the volume for can be complicated. Thus, we need as many tools as possible.



$$V = 2\pi r h \, \Delta r$$

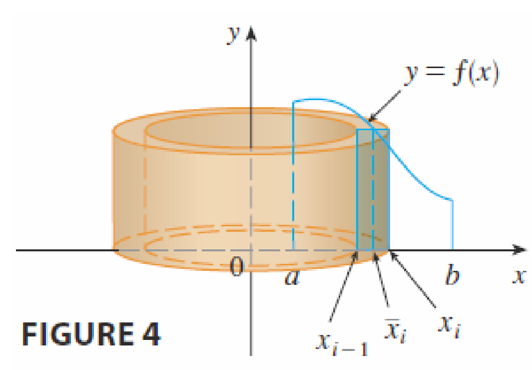
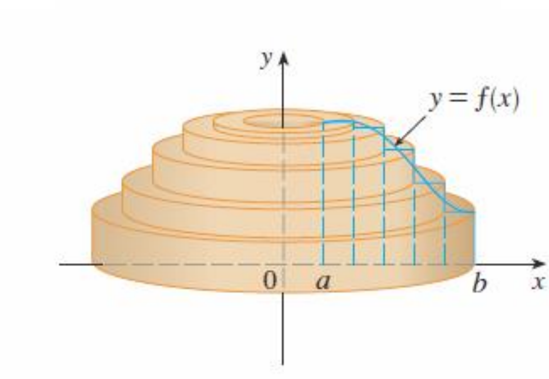
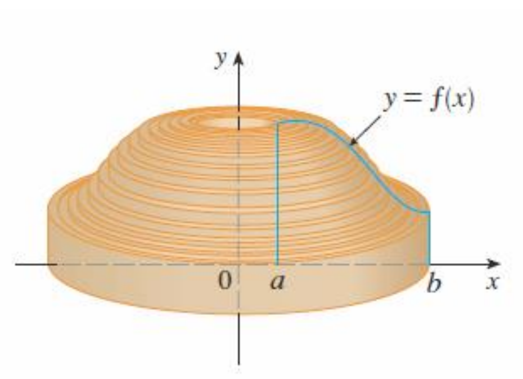
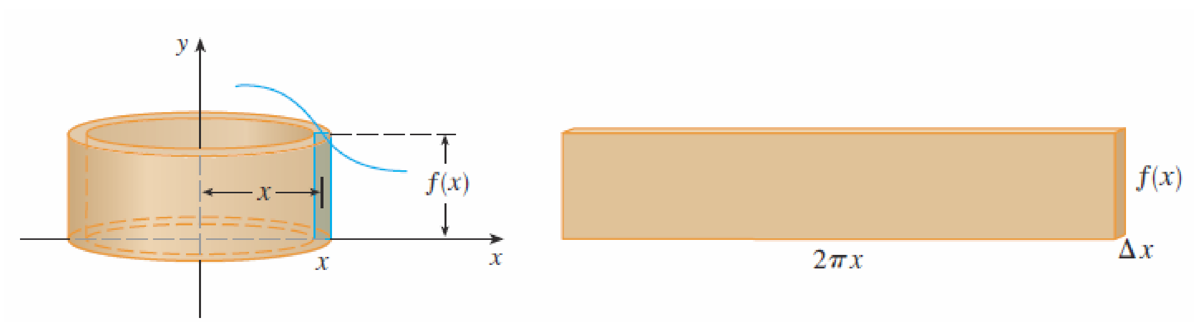


FIGURE 4







$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \; \underbrace{[f(x)]}_{\text{height}} \; \underbrace{dx}_{\text{thickness}}$$

Example 1: Find the volume of the region bound by $y = 2x^2 - x^3$ and $y = 0$ revolved round the y-axis.

Example 2: Find the volume of the region bound by $y = x^3$**,** $x = 0$ and $y = 4 + 2x$ rotated around the y axis.

Example 3: Find the volume of the region bounded by the graph of $y = \sin x$, the x-axis and $0 \le x \le \pi$ rotated about the line $x = -2$.

Example 4: Find the volume of the region R bound by $y = x + 1$ and $y = -\frac{1}{3}x^2 + 1$, (a) rotated around the line $x = 1$ and (b) $y = 2$. Use either the disk or shell method.

Question: How do the limits of integration change roles with the disk/washer method versus the cylindrical shells method?

Disks and Washers versus Cylindrical Shells

When computing the volume of a solid of revolution, how do we know whether to use disks (or washers) or cylindrical shells? There are several considerations to take into account: Is the region more easily described by top and bottom boundary curves of the form $y = f(x)$, or by left and right boundaries $x = g(y)$? Which choice is easier to work with? Are the limits of integration easier to find for one variable versus the other? Does the region require two separate integrals when using x as the variable but only one integral in y? Are we able to evaluate the integral we set up with our choice of variable?

If we decide that one variable is easier to work with than the other, then this dictates which method to use. Draw a sample rectangle in the region, corresponding to a cross-section of the solid. The thickness of the rectangle, either Δx or Δy, corresponds to the integration variable. If you imagine the rectangle revolving, it becomes either a disk (washer) or a shell.

(HANDOUT)

Big Questions for §6.2 & §6.3

1. Where is the axis of rotation with respect to the region?
2. How is the region defined? Function of x? function of y?
3. Is the region defined in parts? That is, does it cross itself?
4. Which radius is the outer-0radius vs. the inner-radius? What is the top and bottom of the shell?

Math 31 | §6.4 Work

Work is a measure of the Force (F) exerted through a distance (d).

$$W = F \cdot d$$

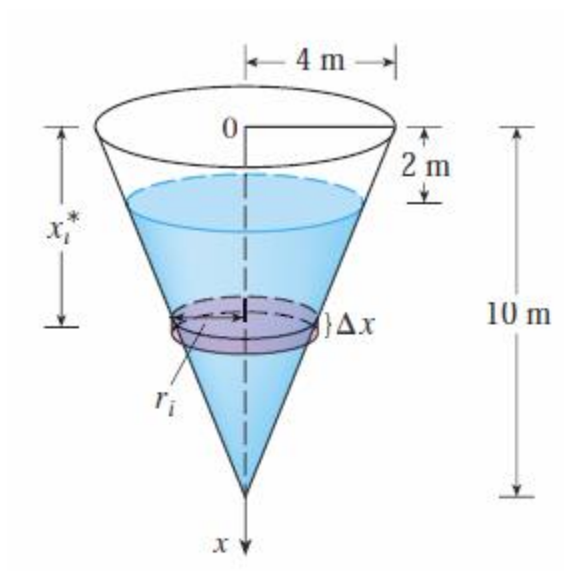
We do not need calculus if the force exerted is constant.

Example 1: Find the work if a force of 3 N is exerted pushing a box for 8 meters.

However if the amount of force is changing over the distance, then calculus may be needed.

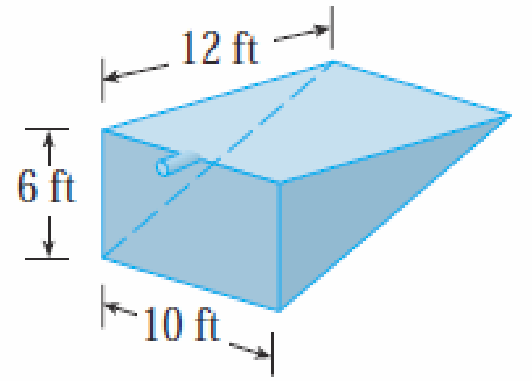
Example 2: Find the work if a varying force of $F(x) = \dfrac{1}{x+1}$ N is applied to a box for x meters for a total distance of 8 meters.

Example 3: A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank.



What do we need to know to answer this question?

Example 4: A tank has the shape of wedge (shown bellow) is completely filled with water. The height of the tank 6 ft., the width of the tank is 10 ft. and the tank extends back 12 ft. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank.



$\approx 62.5 \frac{ft}{sec}$

§6.5 Average Value of a Function

Example 1: Find the average force exerted pushing a box with a force 3 N for 8 meters.

Average Value of a Function over an interval

The average value of f on the interval [a, b] is defined by

$$f_{AVG} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example 2: Find the average force exerted pushing a box of $F(x) = \frac{1}{x+1}$ N for a total distance of 8 meters.

Example 3: Find the average value of the function $f(x) = 188\sin(377x)$ on the interval $[0,1]$

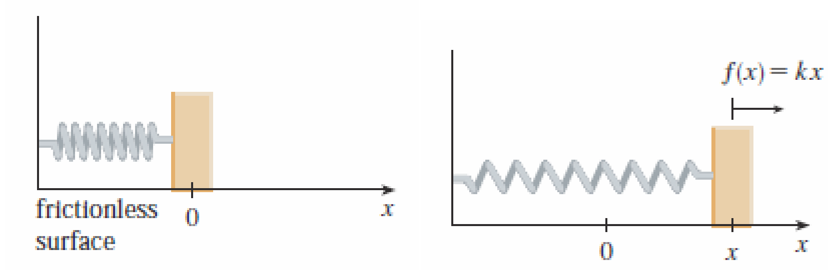
Example 4: In a certain city the temperature (in F^0) t hours after 9 am was modeled by the function, $T(t) = 50 + 12\sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9am to 9 pm.

Hooke's Law (from §6.4)

In the next example we use a law from physics: **Hooke's Law** states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x:

$$f(x) = kx$$

where k is a positive constant (called the **spring constant**). Hooke's Law holds provided that x is not too large (see Figure 1).



(a) Natural position of spring (b) Stretched position of spring

Example 5: A force of 30 N is required to hold a spring stretched from its natural length of 10 cm to 25 cm. How much work is done to stretch the spring from 15 cm to 30 cm?

Example 6: A force of 30 N is required to hold a spring stretched from its natural length of 10 cm to 25 cm. What is the average forced exerted if the spring is stretched from 15 cm to 30 cm?