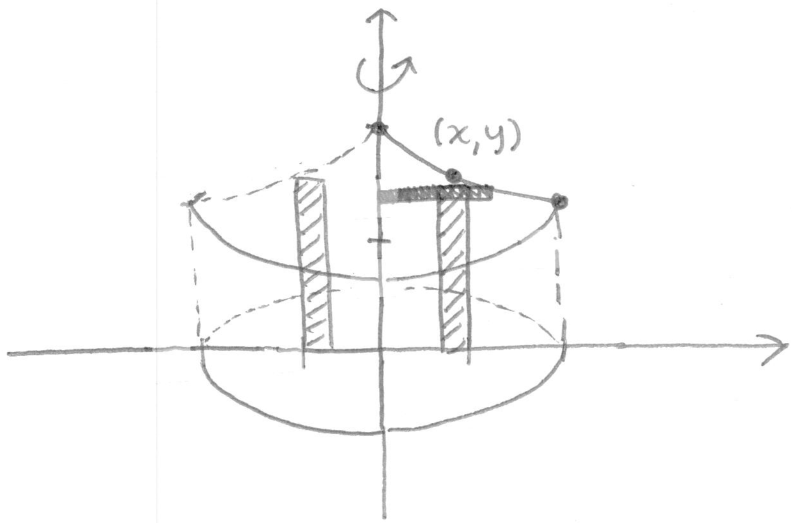
Math 31 | Exam 2 Applications of Integration Spring 2019

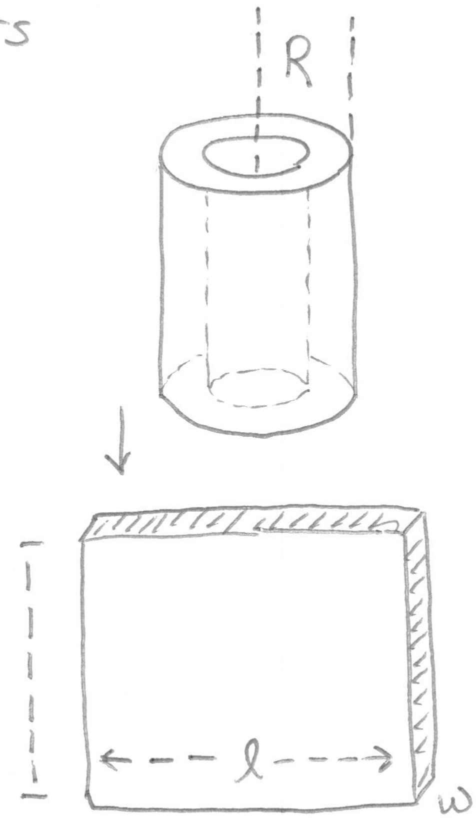
1. [15 PTS] Set up **BUT DO NO EVALUATE** the integral for the volume of the solid of revolution given by the bounded region

$y = 1 + e^{-x}$
$y - 1 = e^{-x}$
$\ln(y-1) = -x$
$x = -\ln(y-1)$
$2 \le y \le 1 + \frac{1}{e^2}$

$$y \le 1 + e^{-x}$$
$$0 \le x \le 2$$

revolved around the line $x = 0$. Draw the solid. 5PTS





CYLINDRKAL SHELLS

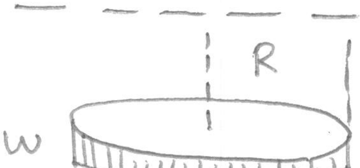
$$V = 2\pi \int_0^2 x(1+e^{-x})\,dx$$

$$V = lwh$$
$$= 2\pi R f(x)\,dx$$
$$= 2\pi x(1+e^{-x})\,dx$$

DISK METHOD

$$V = \pi \int_2^{\frac{e^2+1}{e^2}} (-\ln(y-1))^2\,dy$$

$$= \pi \int_0^2 (1+e^{-x})^2\,dx$$



$$V = \pi R^2 w$$
$$= \pi(-\ln(y-1))^2\,dy$$

CONFUSING METHODS −10

2. [15 PTS] Set up **BUT DO NO EVALUATE** the integral for the volume of the solid of revolution given by the bounded region

$$y = x - 3$$
$$x = -y^2 - 2y + 3 = -(y^2+2y-3) = -(y+3)(y-1)$$

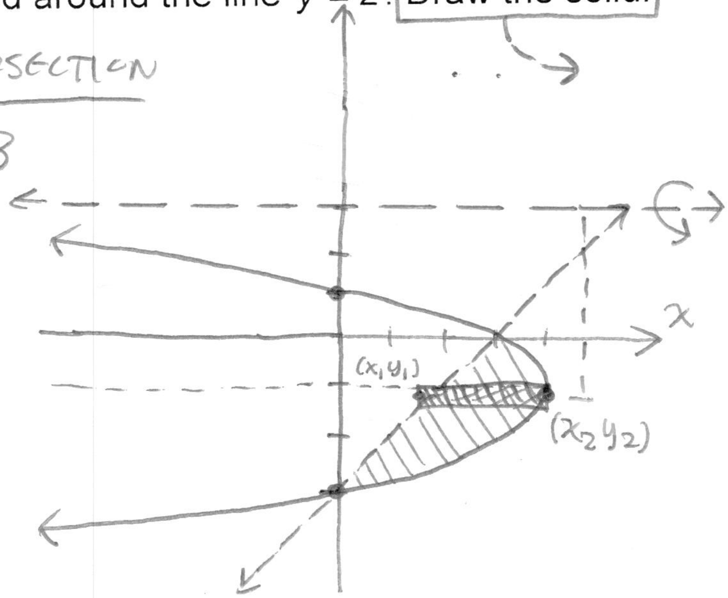
revolved around the line $y = 2$. Draw the solid.

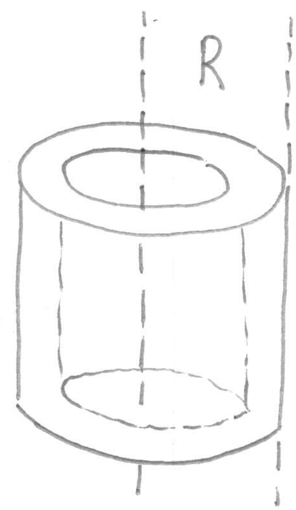
POINTS OF INTERSECTION

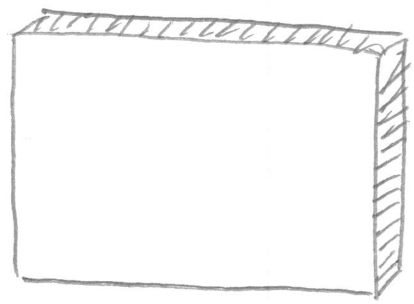
$y+3 = -y^2 - 2y + 3$

$y^2 + 2y = 0$

$y(y+2) = 0$







$$\boxed{V = 2\pi \int_{-3}^{0} (2-y)(-y^2-2y+3-(y+3))\,dy}$$

$V = lwh$

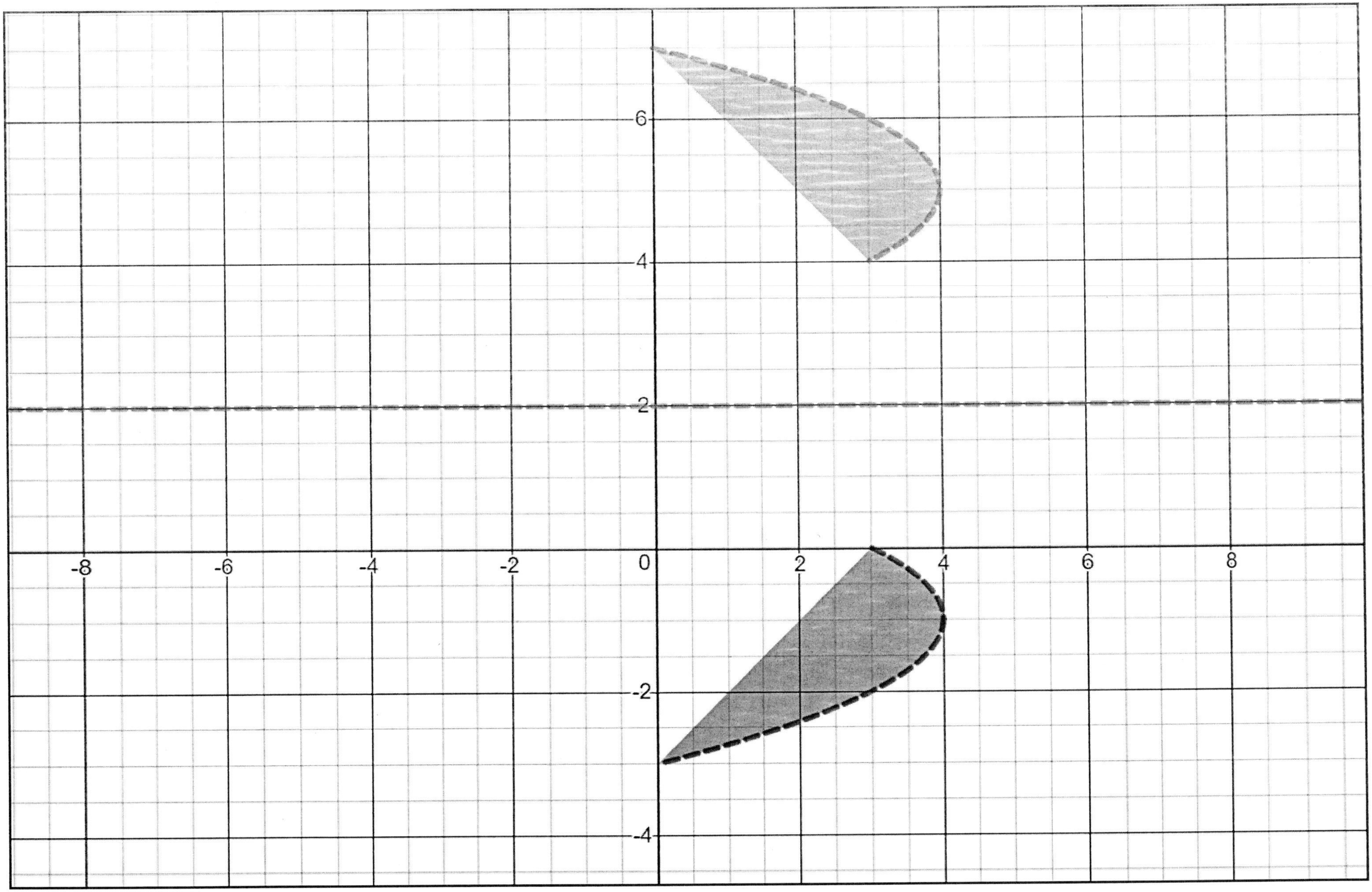
$= 2\pi R(x_2 - x_1)\,dy$

$= 2\pi(2-(y))(-y^2-2y+3-(y-3))\,dy$

$\boxed{\text{AXIS}}$ 4PTS

$\boxed{\text{LIMITS}}$ 3PTS

$\boxed{\text{MIXING METHODS}}$

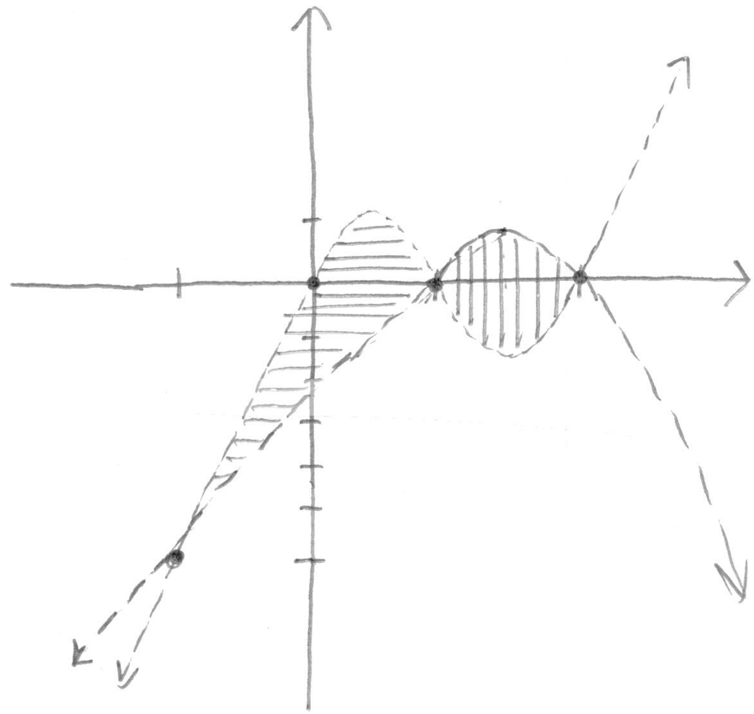


6
4
2
0
-2
-4
-8
-6
-4
-2
2
4
6
8

✓ 3. [15 PTS] Set up, **BUT DO NO EVALUATE** the integral for the area bound by the graphs $y = x^3 - 3x^2 + 2x$ and $y = -x^2 + 3x - 2$.

POINTS OF INTERSECTION

$$x^3 - 3x^2 + 2x = -x^2 + 3x - 2$$
$$x^3 - 2x^2 - x + 2 = 0$$
$$x^2(x-2) - 1(x-2) = 0$$
$$(x-1)(x+1)(x-2) = 0$$



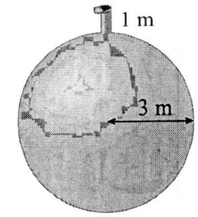
$$y_1 = x^3 - 3x^2 + 2x$$
$$= x(x^2 - 3x + 2)$$
$$= x(x-2)(x-1)$$
$$y_2 = -(x-2)(x-1)$$

$$A = \int_{-1}^{1} (x^3 - 3x^2 + 2x) - (-x^2 + 3x - 2)\, dx$$
$$+ \int_{1}^{2} (-x^2 + 3x - 2) - (x^3 - 3x^2 + 2x)\, dx$$

$\boxed{\text{LIMITS}}$ 8PTS

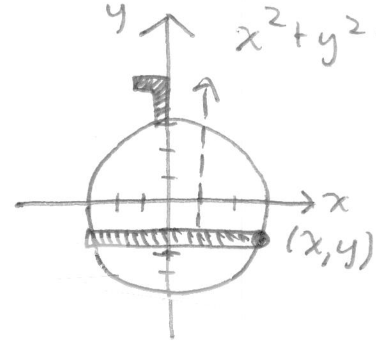
$\boxed{\text{INTEGRAL}}$ 7PTS

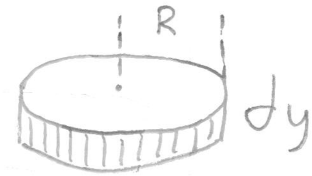
4. [15 PTS] Set up, **BUT DO NO EVALUATE** the integral required to pump all of the water out of a spout one meter above a full spherical tank with a radius of 3 meters (see figure bellow)



$$W = FD$$
$$= maD$$
$$= \rho V g D$$
$$dW = \rho g \pi R^2 D \, dy$$
$$dW = \rho g \pi (9-y^2)(4-y)\, dy$$

$x^2+y^2=9$





$$W = \int_{-3}^{3} dW$$

$$\boxed{W = \rho g \pi \int_{-3}^{3} (9-y^2)(4-y)\, dy}$$

$R = x$

$V = \pi R^2 w$

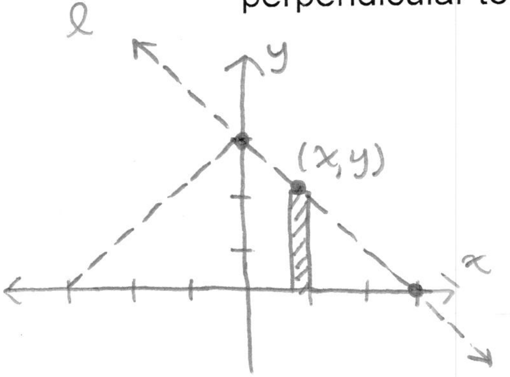
$= \pi x^2 dy$

$= \pi (9-y^2) dy$

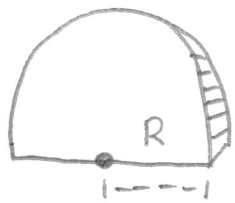
$\boxed{\text{LIMITS}}$

3PTS

5. [15 PTS] Find the volume of a solid that has a triangular base with vertices at (−3,0), (3,3) and (3,0) and semicircular cross sections perpendicular to the x-axis.



$R = \frac{1}{2}y = \frac{3-x}{2}$



$V = \frac{1}{2}AW$

$l: y = 3 - x$

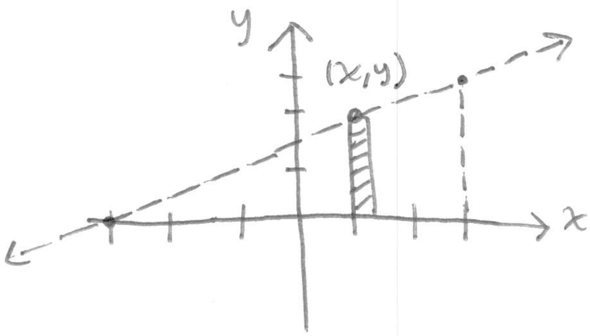
$dV = \frac{1}{2}\pi R^2 dx$

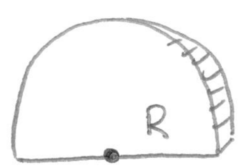
$= \frac{\pi}{2}\left(\frac{3-x}{2}\right)dx$

VOLUME

5 PTS

$$V = \int_{-3}^{3} \frac{\pi}{2}\left(\frac{3-x}{2}\right)^2 dx$$





$R = \frac{y}{2}$

$V = \frac{1}{2}AW$

$l: y = \frac{1}{2}x + b$

$y = \frac{1}{2}x + \frac{3}{2} = \frac{x+3}{2}$

$= \frac{1}{2}\pi R^2 dx$

$= \frac{1}{2}\pi\left(\frac{y}{2}\right)^2 dx = \frac{\pi}{8}y^2 dy$

$$\boxed{V = \int_{-3}^{3} \frac{\pi}{8}\left(\frac{x+3}{2}\right)^2 dx} = \frac{\pi}{32}\left[\int_{-3}^{3} (x+3)^2 dx\right]$$

$$= \frac{\pi}{32}\frac{2}{1}\frac{1}{3}(x+3)^3\Big|_0^3 = \frac{\pi}{96}6^3 = \boxed{\frac{9\pi}{4}}$$

6. [10 PTS] Find the arc length the curve $y = \ln|\sec x|$ for $0 \le x \le \frac{\pi}{4}$. ~~around the y-axis~~.

$$y = \ln|\sec x|$$

$$y' = \frac{1}{\sec x} \cdot \frac{\sec x \tan x}{1}$$

$$= \tan x$$

$$1 + (y')^2 = 1 + \tan^2 x$$

$$= \sec^2 x$$

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x\, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln|\sqrt{2} + 1| - \ln|1|$$

$$= \boxed{\ln(\sqrt{2} + 1)}$$

ASIDE

$$\int \sec\theta\, d\theta$$

$$= \int \frac{\cos\theta}{\cos^2\theta}\, d\theta$$

$$= \int \frac{\cos\theta}{1 - \sin^2\theta}\, d\theta$$

$$= \int \frac{1}{1 - u^2}\, d\theta$$

$$= \frac{1}{2} \int \frac{1}{1-u} + \frac{1}{1+u}\, du$$

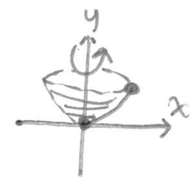
$$= \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + C$$

$$= \frac{1}{2} \ln\left|\frac{(1+u)^2}{1-u^2}\right| + C$$

$$= \frac{1}{2} \ln\left|\frac{(1+\sin\theta)^2}{\cos^2\theta}\right| + C$$

$$= \ln|\sec\theta + \tan\theta| + C$$

7. [10 PTS] Find the surface area of revolving the curve $y = \frac{1}{3}x^{\frac{3}{2}}$ for $0 \le x \le 1$ around the y–axis.



$$S = 2\pi \int_a^b x\sqrt{1+f'(x)^2}\,dx$$

$$f(x) = \frac{1}{3}x^{3/2}$$

$$f'(x) = \frac{1}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} = \frac{1}{2}x^{\frac{1}{2}}$$

$$1+(f'(x))^2$$
$$= 1+\frac{1}{4}x$$

$$S = 2\pi\int_0^1 x\sqrt{1+\frac{1}{4}x}\,dx$$

ASIDE

$$u = \sqrt{1+\frac{1}{4}x}$$
$$u^2 = 1+\frac{x}{4}$$
$$2u\,du = \frac{1}{4}dx$$
$$4(u^2-1) = x$$
$$u(0) = \sqrt{1} = 1$$
$$u(1) = \frac{\sqrt{5}}{2}$$

$$= 2\pi\int_1^{\frac{\sqrt{5}}{2}} 4(u^2-1)\,u\,(8u)\,du$$

$$= 64\pi\int_1^{\frac{\sqrt{5}}{2}} u^4 - u^2\,du$$

$$= 64\pi\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]\Bigg|_1^{\frac{\sqrt{5}}{2}}$$

$$= 64\pi\left[\frac{1}{5}\left(\frac{\sqrt{5}}{2}\right)^5 - \frac{1}{3}\left(\frac{\sqrt{5}}{2}\right)^2 - \frac{1}{5} + \frac{1}{3}\right]$$

$$= 64\pi\left[\frac{25\sqrt{5}}{\cancel{8}\cdot 2} - \frac{5}{6} + \frac{2}{15}\right] = \left[\frac{75\sqrt{5}-25+4}{30}\right]\frac{64\pi}{1}$$

$$= 64\pi\left[\frac{75\sqrt{5}-21}{30}\right]$$

$$= \boxed{\frac{32\pi(25\sqrt{5}-7)}{5}}$$

$$S = 2\pi \int_0^1 x\sqrt{1+\frac{x}{4}}\, dx$$

$$= 2\pi \int_0^1 \frac{x\sqrt{4+x}}{\sqrt{4}}\, dx$$

$$= \pi \int_0^1 x\sqrt{4+x}\, dx$$

$u = 4 + x \quad x = u - 4$

$du = dx$

$$= \pi \int_4^5 (u-4)\sqrt{u}\, du$$

$$= \pi \int_4^5 u^{3/2} - 4u^{\frac{1}{2}}\, du$$

$$= \pi \left[\frac{2}{5}u^{5/2} - \frac{4}{1}\cdot\frac{2}{3}u^{3/2}\right]_4^5$$

$$= \pi \left[\frac{2}{5}5^{5/2} - \frac{8}{3}5^{3/2} - \frac{2}{5}4^{5/2} + \frac{8}{3}4^{3/2}\right]$$

$$= \pi\left[5\sqrt{5}\left(2 - \frac{8}{3}\right) - 4\sqrt{4}\left(\frac{8}{5} - \frac{8}{3}\right)\right] =$$

$$= \pi\left(5\sqrt{5}\left(-\frac{2}{3}\right) - 64\left(-\frac{2}{15}\right)\right)$$

$$= \pi\left[\frac{-10\sqrt{5}}{3} + \frac{128}{15}\right]$$

$$= \pi\left[\frac{128 - 50\sqrt{5}}{15}\right]$$

YOUR INPUT:
$f(x) =$

$$2\pi\sqrt{\frac{x}{4}+1x}$$

Note: Your input has been rewritten/simplified.

Simplify/rewrite:

$$\pi x\sqrt{x+4}$$

"MANUALLY" COMPUTED ANTIDERIVATIVE:
$\int f(x)\,\mathrm{d}x = F^{*}(x) =$

"Manual" integration with steps:
The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might only be valid for parts of the function.

$$\frac{2\pi(x+4)^{\frac{3}{2}}(3x-8)}{15}+C$$

Hide steps

$$\frac{2\pi(x+4)^{\frac{3}{2}}(3x-8)}{15}+C$$

Hide steps

Problem:

$$\int 2\pi\sqrt{\frac{x}{4}+1x}\,\mathrm{d}x$$

Apply linearity:

$$=\pi\int x\sqrt{x+4}\,\mathrm{d}x$$

Now solving:

$$\int x\sqrt{x+4}\,\mathrm{d}x$$

Substitute $u = x+4 \longrightarrow \mathrm{d}x = \mathrm{d}u$ (steps):

$$=\int\left(u^{\frac{3}{2}}-4\sqrt{u}\right)\mathrm{d}u$$

Apply linearity:

$$=\int u^{\frac{3}{2}}\,\mathrm{d}u-4\int\sqrt{u}\,\mathrm{d}u$$

Now solving:

$$\int u^{\frac{3}{2}}\,\mathrm{d}u$$

Apply power rule:

$$\int u^{\mathrm{n}}\,\mathrm{d}u=\frac{u^{\mathrm{n}+1}}{\mathrm{n}+1}\text{ with }\mathrm{n}=\frac{3}{2}:$$

$$= \frac{2u^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\int u^{\frac{3}{2}}\,\mathrm{d}u - 4\int \sqrt{u}\,\mathrm{d}u$$

$$= \frac{2u^{\frac{5}{2}}}{5} - \frac{8u^{\frac{3}{2}}}{3}$$

Undo substitution $u = x + 4$:

$$= \frac{2(x+4)^{\frac{5}{2}}}{5} - \frac{8(x+4)^{\frac{3}{2}}}{3}$$

Plug in solved integrals:

$$\pi \int x\sqrt{x+4}\,\mathrm{d}x$$

$$= \frac{2\pi(x+4)^{\frac{5}{2}}}{5} - \frac{8\pi(x+4)^{\frac{3}{2}}}{3}$$

The problem is solved:

$$\int 2\pi\sqrt{\frac{x}{4} + 1}x\,\mathrm{d}x$$

$$= \frac{2\pi(x+4)^{\frac{5}{2}}}{5} - \frac{8\pi(x+4)^{\frac{3}{2}}}{3} + C$$

Rewrite/simplify:

$$= \frac{2\pi(x+4)^{\frac{3}{2}}(3x-8)}{15} + C$$

ANTIDERIVATIVE COMPUTED BY MAXIMA:

$\int f(x)\,\mathrm{d}x = F(x) =$

$$\pi\left(\frac{2(x+4)^{\frac{5}{2}}}{5} - \frac{8(x+4)^{\frac{3}{2}}}{3}\right) + C$$

Simplify/rewrite:

$$\frac{2\pi(x+4)^{\frac{3}{2}}(3x-8)}{15} + C$$

DEFINITE INTEGRAL:

$\int_0^1 f(x)\,\mathrm{d}x =$

$$\left(\frac{128}{15} - \frac{2\cdot 5^{\frac{3}{2}}}{3}\right)\pi$$

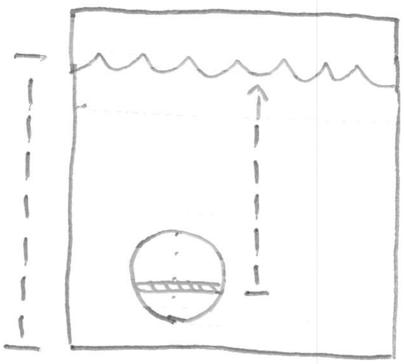
Simplify/rewrite:

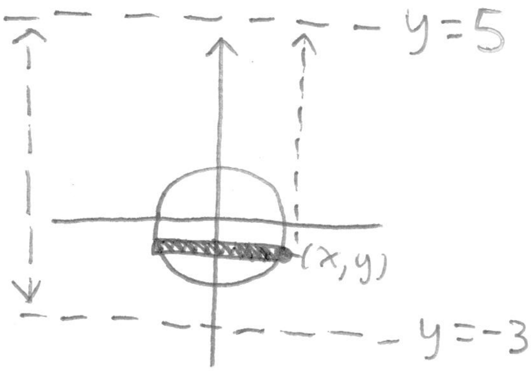
$$-\frac{2\left(5^{\frac{5}{2}} - 64\right)\pi}{15}$$

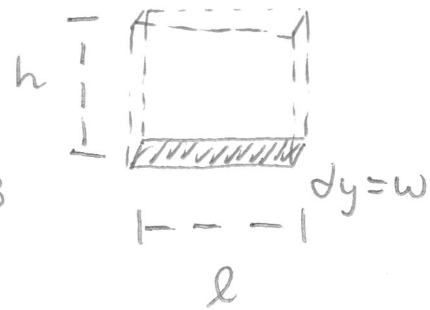
Approximation:

3.392208207163814

8. [15 PTS] Set up, **BUT DO NO EVALUATE** the integral the represents the total hydrostatic force on the viewing window. A circular viewing window of radius 2 meters sits 1 meter from the bottom of a 10-meter deep tank of water. If the depth of the water is 8 meters in the tank, find the total hydrostatic force on the viewing window.







$x^2 + y^2 = 4$

$x = \sqrt{4-y^2}$

$V = \ell w h$

$= 2x(dy)(5-y)$

$= 2x(5-y)dy$

$F = ma$

$= \rho V g$

$= (5-y)\rho g \, dA$

$= (5-y)\rho g (2x)\, dy$

$= \rho g (5-y)(2(\sqrt{4-y^2}))dy$

$$\boxed{F = 2\rho g \int_{-2}^{2} (5-y)\sqrt{4-y^2}\, dy}$$

LIMITS
4 PTS