Math 31 | Exam 1 SOLUTIONS | Spring 2018

1. $\int_0^2 \sqrt{4x - x^2}\, dx$

PROBLEM 14-9 Find $\int_0^2 \sqrt{4x - x^2}\, dx$.

Solution: Complete the square, then use an inverse trigonometric substitution:

$$\int_0^2 \sqrt{4x - x^2}\, dx = \int_0^2 \sqrt{4 - (4 - 4x + x^2)}\, dx$$
$$= \int_0^2 \sqrt{4 - (2 - x)^2}\, dx \checkmark$$

The radicand is in the form $a^2 - u^2$; you substitute $u = a \sin\theta$:

$$2 - x = 2\sin\theta$$
$$dx = -2\cos\theta\, d\theta \checkmark$$

$\checkmark$

To change limits: If $x = 0$, $\sin\theta = 1$ and $\theta = \pi/2$. If $x = 2$, $\sin\theta = 0$ and $\theta = 0$. You get

$$\int_0^2 \sqrt{4 - (2 - x)^2}\, dx = -\int_{\pi/2}^0 \sqrt{4 - 4\sin^2\theta}\, 2\cos\theta\, d\theta = 4\int_0^{\pi/2} \cos^2\theta\, d\theta$$
$$\checkmark = \frac{4}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]\Bigg|_0^{\pi/2}$$
$$= 2\left[\frac{\pi}{2} + \frac{\sin\pi}{2}\right] = \pi \checkmark$$

[See Section 14-3.]

+

2. $\int_1^{e^\pi} \cos(\ln x)\,dx$

PROBLEM 14-11 • Find $\int_1^{e^\pi} \cos(\ln x)\,dx.$

Solution: Integrate by parts:

$$u = \cos(\ln x),\ du = -\sin(\ln x)\frac{1}{x}\,dx \qquad dv = dx,\ v = x$$

$$\int_1^{e^\pi} \cos(\ln x)\,dx = x\cos(\ln x)\Big|_1^{e^\pi} + \int_1^{e^\pi} x\sin(\ln x)\frac{dx}{x}$$

$$= e^\pi\cos(\ln e^\pi) - 1\cos(\ln 1) + \int_1^{e^\pi} \sin(\ln x)\,dx$$

Integrate by parts, again, with

$$u = \sin(\ln x),\ du = \frac{\cos(\ln x)}{x} \qquad dv = dx,\ v = x$$

$$\int_1^{e^\pi} \cos(\ln x)\,dx = e^\pi\cos\pi - 1\cos 0 + \left[x\sin(\ln x)\Big|_1^{e^\pi} - \int_1^{e^\pi}\cos(\ln x)\,dx\right]$$

Add $\int_1^{e^\pi} \cos(\ln x)\,dx$ to both sides and divide by two:

$$\int_1^{e^\pi} \cos(\ln x)\,dx = \frac{1}{2}[-e^\pi - 1 + (e^\pi\sin\pi - 1\sin 0)] = -\frac{1}{2}(e^\pi + 1)$$

[See Section 14-1.]

3. $\int \frac{x^4+9x^2+15}{x^5-x^4+8x^3-8x^2+16x-16}\,dx = \int \frac{x^4+9x^2+15}{(x-1)(x^2+4)^2}\,dx$

[HCB.14.43]

ASIDE

$$\frac{x^4+9x^2+15}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+F}{(x^2+4)^2}$$

$$x^4+9x^2+15 = A(x^2+4)^2 + (Bx+c)\underbrace{(x^2+4)(x-1)}_{(x^3-x^2+4x-4)} + (Dx+F)(x-1)$$

IF $x=1$ THEN

$25 = A(25) \therefore A=1$

$$x^4+9x^2+15 = A(x^4+8x^2+16)$$
$$(Bx+c)(x^3-x^2+4x-4)$$
$$+Dx^2-Dx+Fx-F$$

$A+B=1$ | $8A+D=9$ | $-D+F=0$

$-B+C=0$ | $8(1)+D=9$ | $\therefore F=1$

$\therefore B=C=0$ | $\therefore D=1$

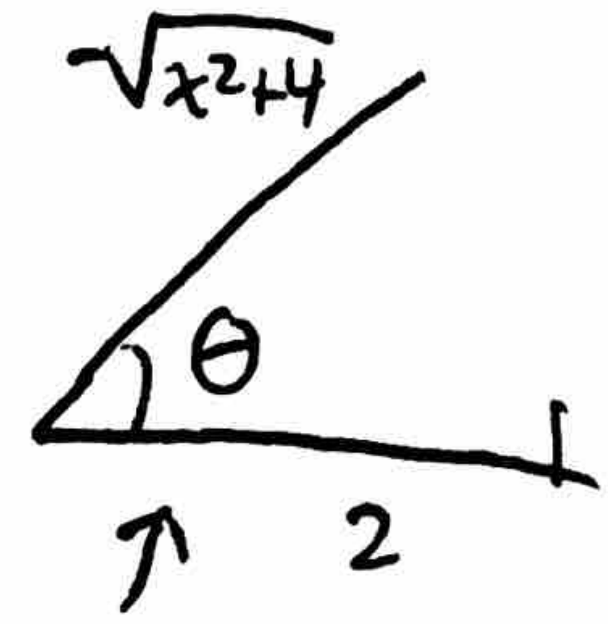
(14-43) (Partial fractions; inverse trigonometric substitution):

$$\ln|x-1| - \frac{1}{2(x^2+4)} + \frac{x}{8(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\boxed{= \ln|x-1| + \frac{x-4}{8(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + c}$$

$$\tan\theta = \frac{x}{2}$$

$$\theta = \tan^{-1}\left(\frac{x}{2}\right)$$



$$= \int \frac{1}{x-1} + \frac{x+1}{(x^2+4)^2}\, dx$$

$$= \ln|x-1| + \frac{1}{2}\int \frac{2x}{(x^2+4)^2}\, dx + \int \frac{1}{(x^2+4)^2}\, dx$$

$u = x^2+4$ $\qquad$ $x = 2\tan\theta$

$du = 2x\, dx$ $\qquad$ $dx = 2\sec^2\theta\, d\theta$

$$= \ln|x-1| - \frac{1}{2}\cdot\frac{1}{x^2+4} + \int \frac{2\sec^2\theta\, d\theta}{16(\tan^2\theta+1)^2}$$

$w = \tan\theta$

$dw = \sec^2\theta\, d\theta$

$$= \ln|x-1| - \frac{1}{2}\cdot\frac{1}{x^2+4} + \frac{1}{8}\int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} - \frac{1}{8}\int \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} + \frac{1}{8}\int \frac{1}{2} + \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} + \frac{1}{16}\theta - \frac{1}{16}\sin 2\theta + C$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{16}\sin\theta\cos\theta + C$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{16}\,\frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}$$

$$= \ln|x-1| - \frac{1}{2(x^2+4)} - \frac{x}{8(x^2+4)} + \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)$$

$$\frac{x^4+9x^2+15}{(x-1)(x^2+4)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} + \frac{Dx+F}{(x^2+4)^2}$$

$$x^4+9x^2+15 = A(x^2+4)^2 + (Bx+C)(x^2+4)(x-1) + (Dx+F)(x-1)$$

$$= A(x^4+8x^2+16) + (Bx+C)(x^3-x^2+4x-4) + (Dx^2-Dx+Fx-F)$$

$$\begin{aligned} x^4+9x^2+15 = \; & Ax^4 && && +8Ax^2 && && +16A \\ & Bx^4 && -Bx^3 && +4Bx^2 && -4Bx && \\ & && +Cx^3 && -Cx^2 && +4Cx && -4C \\ & && && Dx^2 && -Dx && \\ & && && && +Fx && -F \end{aligned}$$

$A+B=1$

$-B+C=0$

$\boxed{A=1}$ BY COVER-UP

$\therefore \boxed{B=C=0}$

$9 = 8A+4B-C+D$

$9 = 8(1)+D$

$\therefore \boxed{D=1}$

$0=-4B+4C-D+F$

$0=-D+F$

$0=-1+F$ $\therefore \boxed{F=1}$

4. $\int \sin(5x)[\sin(7x)+2\sin(5x)]\,dx$

$$= \int \sin(5x)\sin(7x)\,dx + 2\int \sin^2(5x)\,dx$$

$$= \frac{1}{2}\int \cos(7x-5x)\,dx - \frac{1}{2}\int \cos(7x+5x)\,dx + 2\int \frac{1-\cos(10x)}{2}\,dx$$

$$= \frac{1}{2}\int \cos 2x\,dx - \frac{1}{2}\int \cos(12x)\,dx + \int 1-\cos(10x)\,dx$$

$$= \frac{1}{4}\sin 2x - \frac{1}{24}\sin(12x) - \frac{1}{10}\sin(10x) + x + C$$

$$\int \sin(5x)[\sin(7x)+2\sin(5x)]\,dx = \frac{-5\sin(12x)-12\sin(10x)+30\sin(2x)}{120} + x$$

5. $\int \frac{x}{\sqrt{x^2+4x+13}}\,dx$

$$= \int \frac{x}{\sqrt{(x+2)^2+9}}\,dx$$

$$= \int \frac{u-2}{\sqrt{u^2+9}}\,du$$

$$= \int \frac{u}{\sqrt{u^2+9}} - \frac{2}{\sqrt{u^2+9}}\,du$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw - 2\int \frac{1}{\sqrt{u^2+9}}\,du$$

$$-2\int \frac{3\sec^2\theta\,d\theta}{3\sec\theta}$$

$$= \frac{1}{2}\left(2w^{\frac{1}{2}}\right) - 2\int \sec\theta\,d\theta$$

$$= \sqrt{u^2+9} - 2\ln\left|\tan\theta + \sec\theta\right| + C$$

$$= \sqrt{(x+2)^2+9} - 2\ln\left|\frac{x+2}{3} + \frac{\sqrt{(x+2)^2+9}}{3}\right|$$

$$\int \frac{x}{\sqrt{x^2+4x+13}}\,dx = \sqrt{x^2+4x+13} - 2\ln\left(\left|\sqrt{(x+2)^2+9} + x + 2\right|\right) + C$$

$u = x+2$

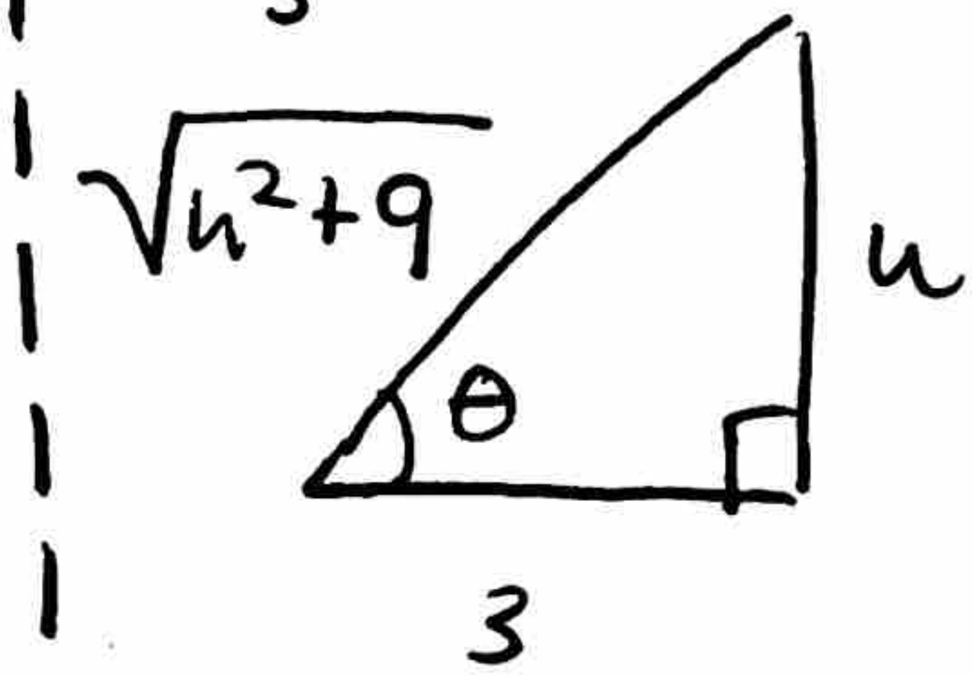
$du = dx$

$x = u-2$

~ [HCB.14.48]

$u = 3\tan\theta$

$\frac{u}{3} = \tan\theta$



$w = u^2+9$

$dw = 2w\,du$

$du = 3\sec^2\theta\,d\theta$

$$\int \sec\theta\,d\theta$$

$$= \int \frac{\cos\theta}{\cos^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta\,d\theta}{1-\sin^2\theta}$$

$$= \int \frac{1}{1-v^2}\,dv$$

$$= \int \frac{\frac{1}{2}}{1-v} + \frac{\frac{1}{2}}{\ldots}\,dv$$

6. $\int_0^4 \frac{1}{x+3\sqrt{x}-4}\,dx$

$$= \int_0^2 \frac{2u}{u^2+3u-4}\,du$$

$$= 2\int_0^2 \frac{A}{u+4} + \frac{B}{u-1}\,du$$

$$= 2\int_0^2 \frac{\frac{4}{5}}{u+4} + \frac{\frac{1}{5}}{u-1}\,du$$

$$= \frac{8}{5}\int_0^2 \frac{1}{u+4}\,du + \frac{2}{5}\int_0^2 \frac{1}{u-1}\,du$$

$$= \frac{8}{5}\ln|u+4| + \frac{2}{5}\ln|u-1|$$

$$= \frac{8}{5}(\ln 6 - \ln 4) + \frac{2}{5}(\ln 1 - \ln|-1|)$$

$$= \frac{8}{5}\ln\left(\frac{3}{2}\right) + \frac{2}{5}(0)$$

$$= \boxed{\frac{8}{5}\ln\left(\frac{3}{2}\right)}$$

$$\approx 0.648744173$$

$u(0)=0$

$u(4)=2$

[HCB.14.32]

$u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$u = A(u-1) + B(u+4)$

IF $u=1$

$1 = 5B \therefore B = \frac{1}{5}$

IF $u=-4$

$-4 = -5A \therefore A = \frac{4}{5}$

(14-32) ($u = \sqrt{x}$, then partial fractions):

$$\int 2u/(u^2 + 3u - 4)\,du = (8/5)\ln|\sqrt{x} + 4| + (2/5)\ln|\sqrt{x} - 1| + C$$