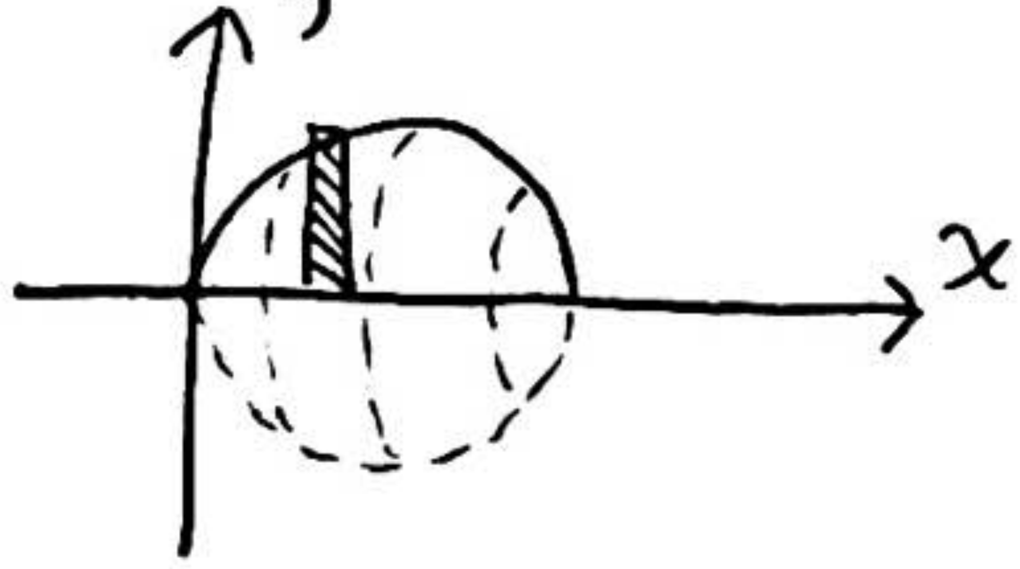
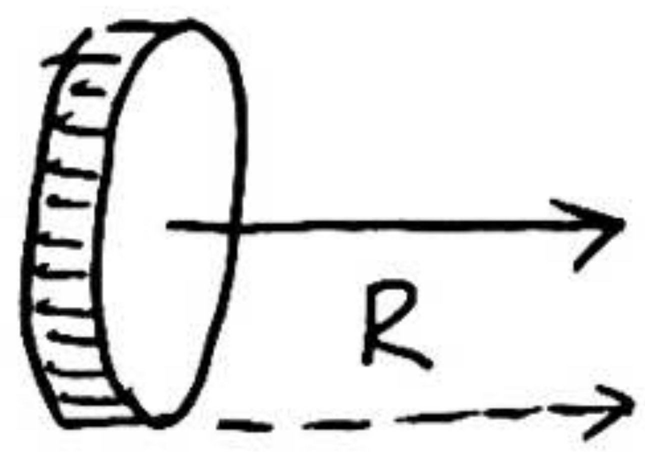
$\boxed{\S 6.2}$

$$y = \sin x$$



$$V = \int_0^{\pi} \pi \sin^2 x \, dx$$



$$= \pi \int_0^{\pi} \left(\frac{1 - \cos 2x}{2} \right) dx$$

$$dV = \pi R^2 \, dx$$

$$dV = \pi (\sin x)^2 \, dx$$

$$= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x \, dx$$

$$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi}$$

$$= \frac{\pi}{2} \left[(\pi - 0) - \frac{1}{2} (\sin 2\pi - \sin 0) \right]$$

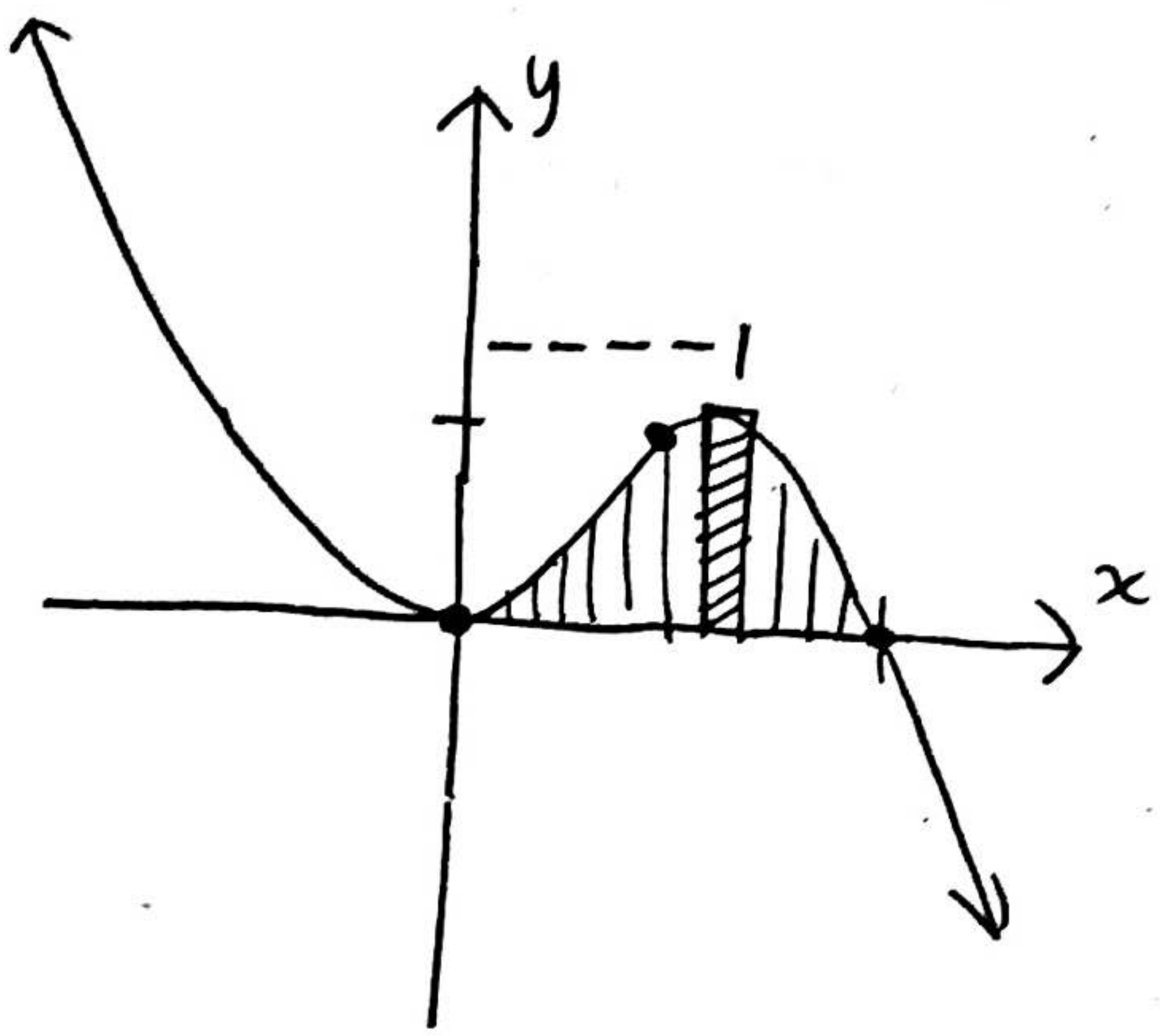
$$= \frac{\pi^2}{2}$$

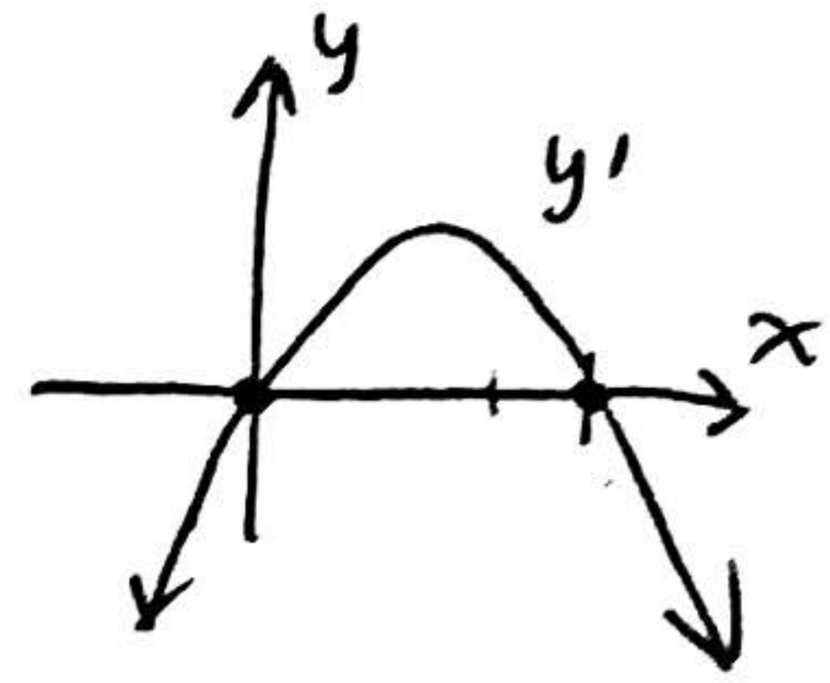
$$\boxed{\text{VOLUME OF LEMON}}$$

E1

$$y = 2x^2 - x^3$$
$$= x^2(2 - x)$$

$$y' = 4x - 3x^2$$
$$= x(4 - 3x)$$



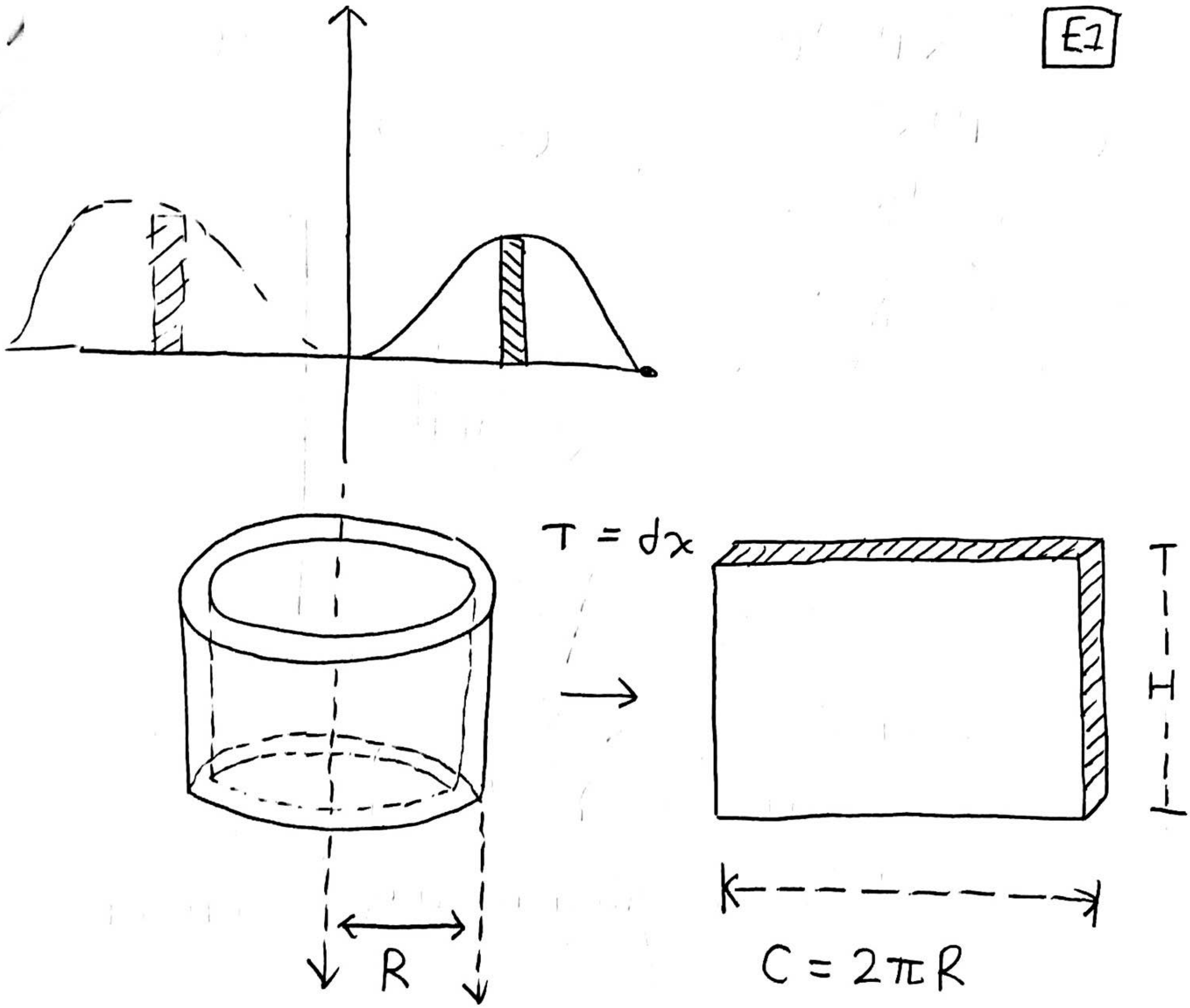


OBVIOUSLY $0 \le x \le 2$ BUT...

IF AXIS IS Y ... THEN WE HAVE dy.

ENTER CYLINDRICAL SHELLS!

E1



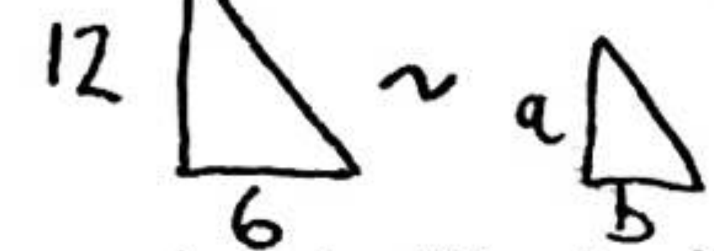
$$V = T \cdot H \cdot C$$

$$dV = 2\pi R(f(x))\,dx$$

$$dV = 2\pi x f(x)\,dx$$

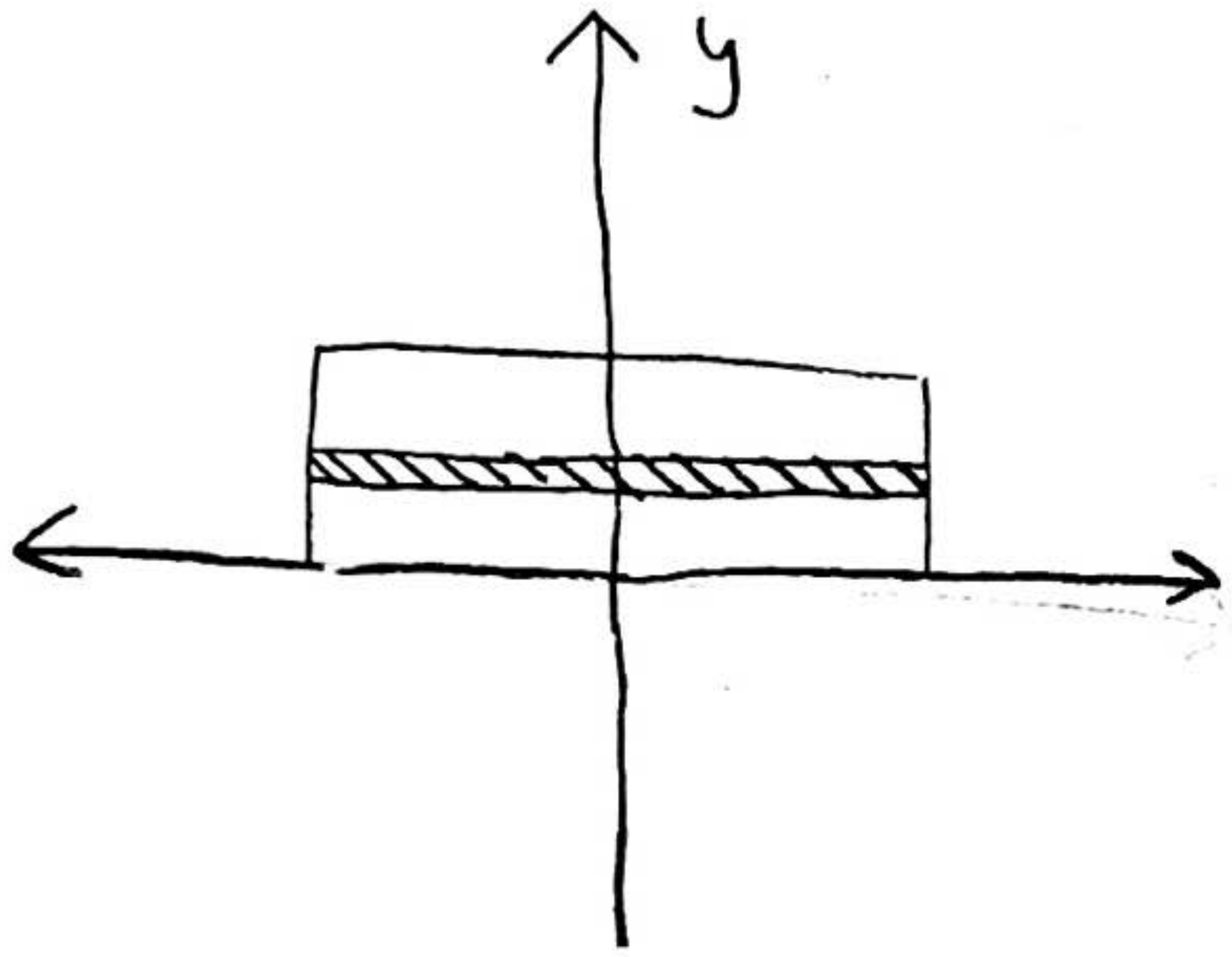
$$V = \int_a^b 2\pi x f(x)\,dx$$

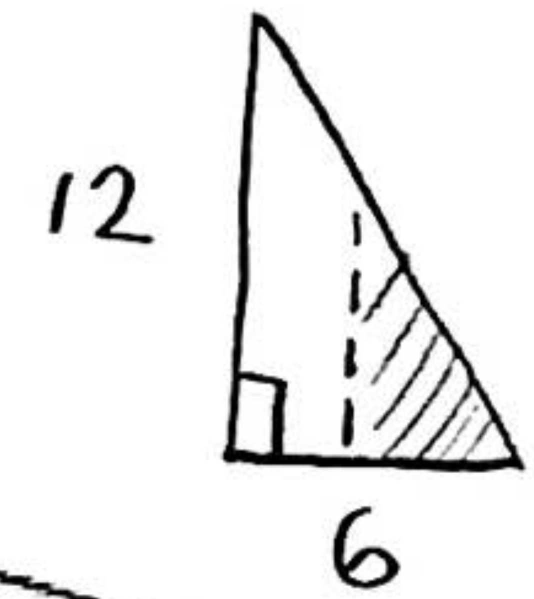
$\frac{12}{a} = \frac{6}{b}$

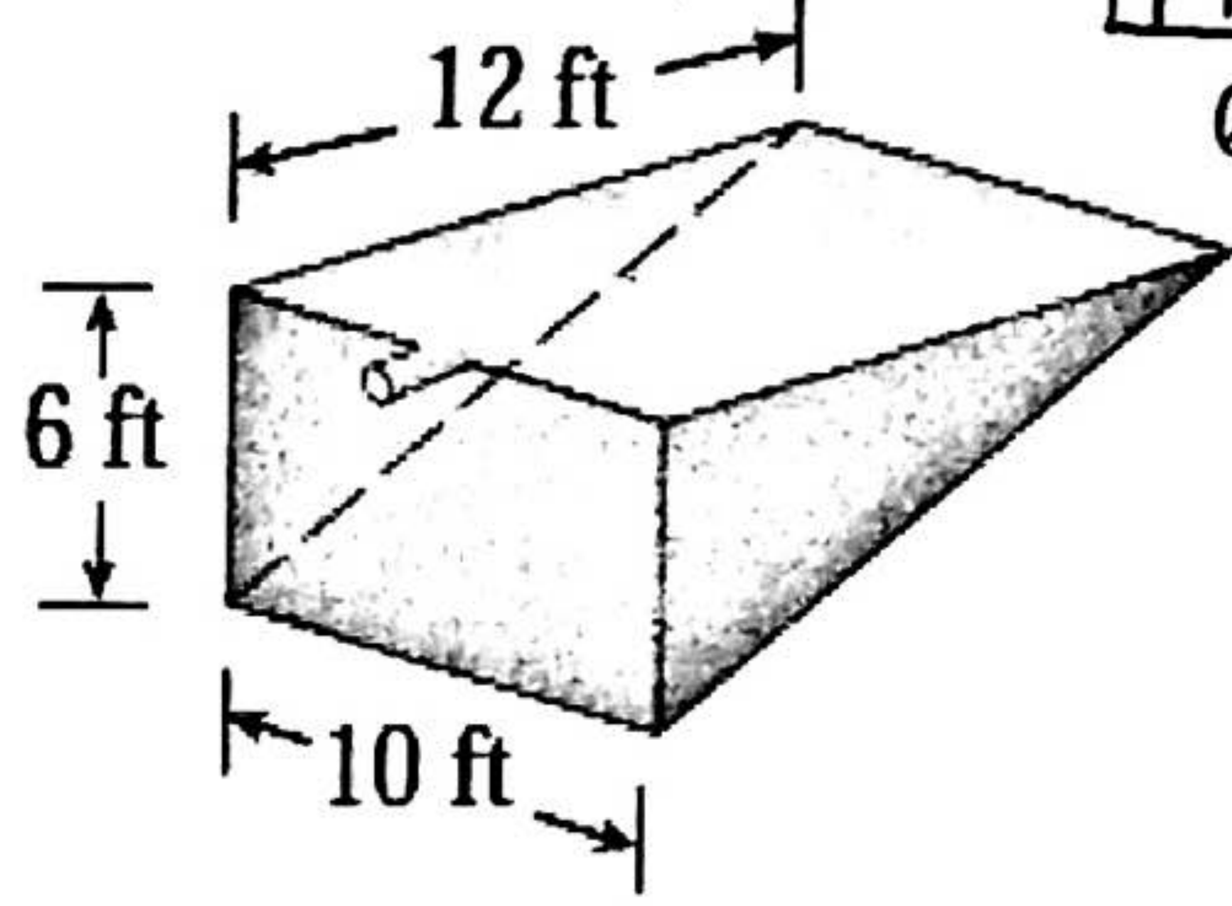


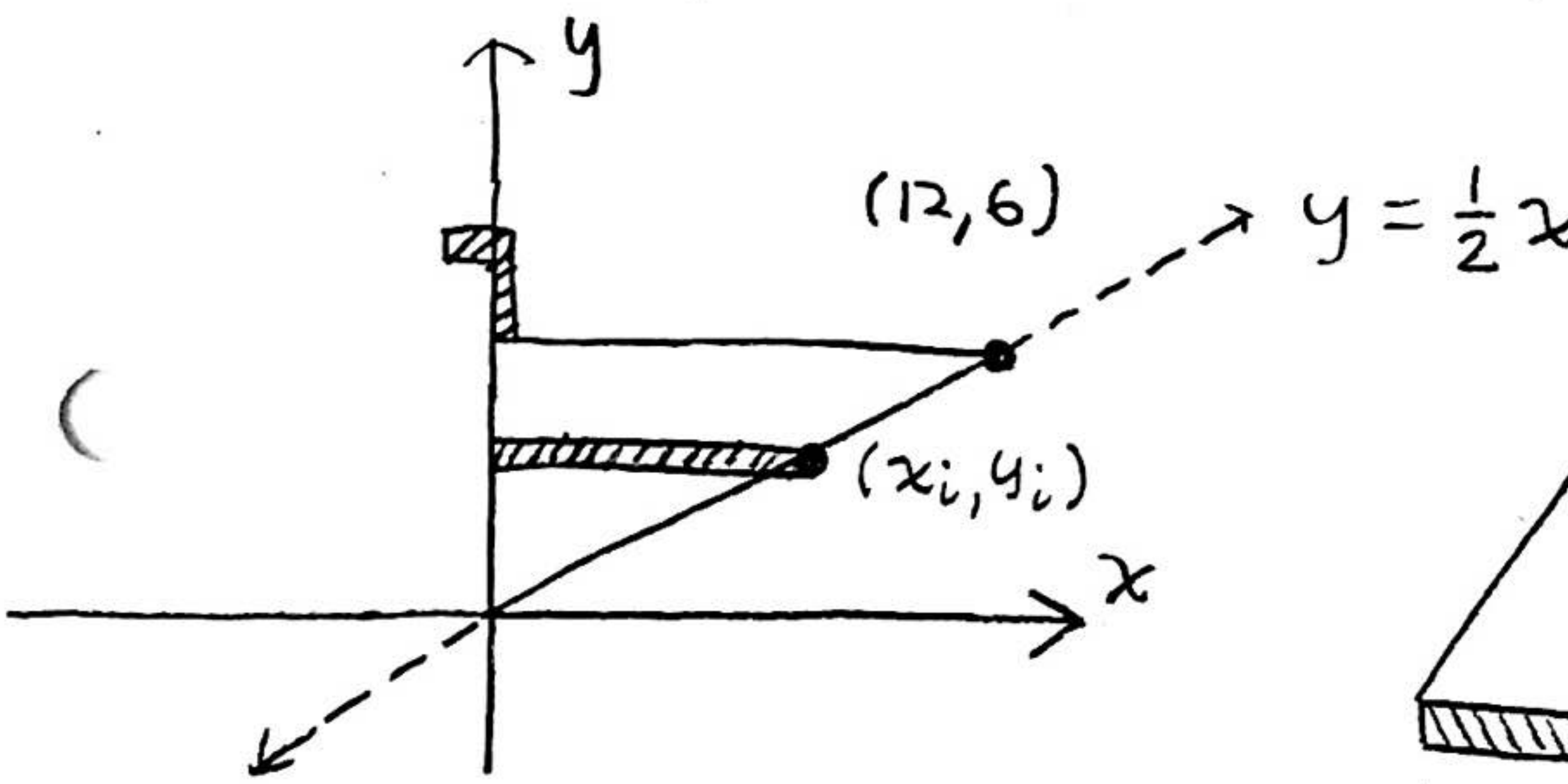
WORK

Example: A tank has the shape of wedge (shown bellow) is completely filled with water. The height of the tank 6 ft., the width of the tank is 10 ft. and the tank extends back 12 ft. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank.

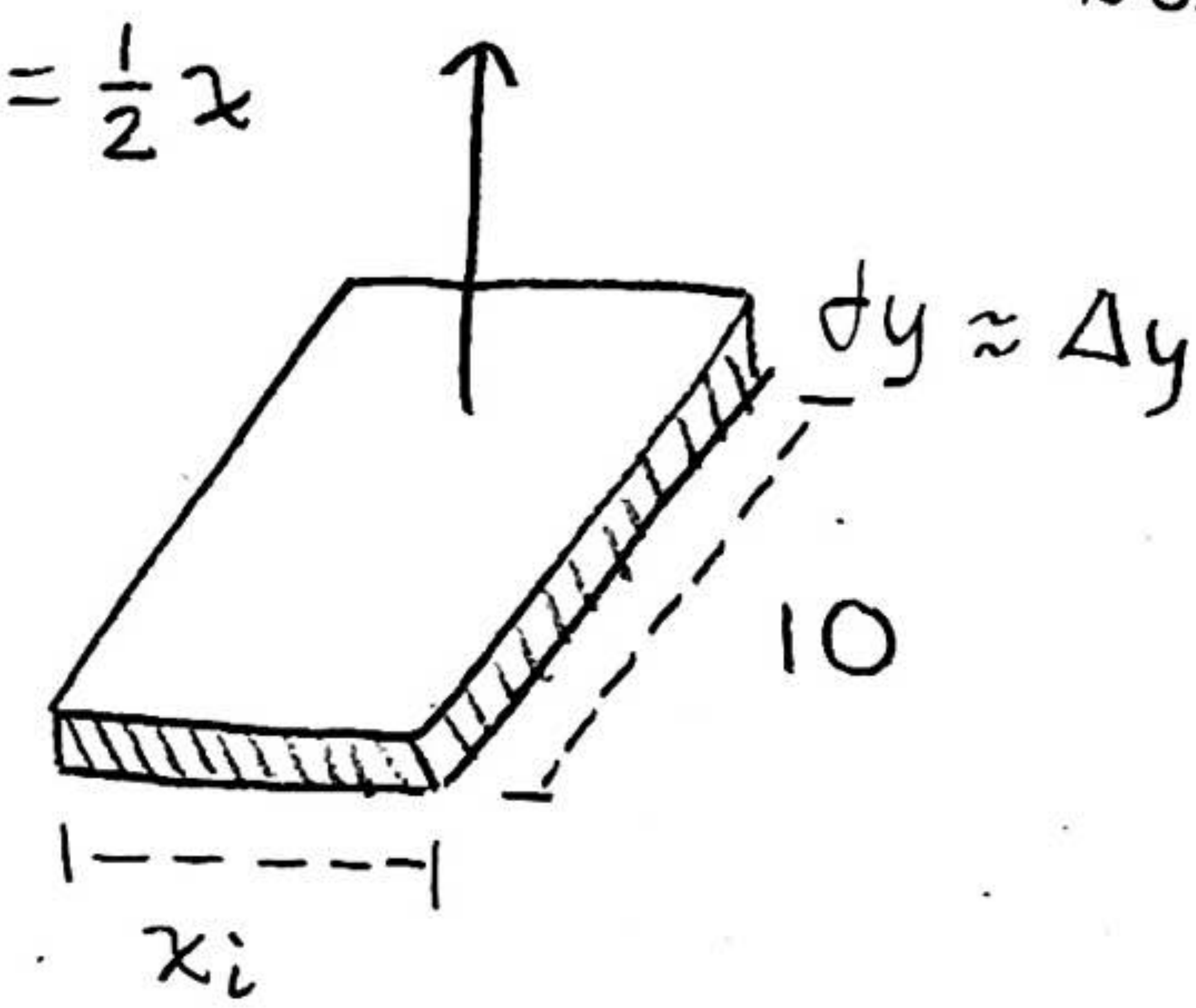








≈ 62.5 lbs/ft^3



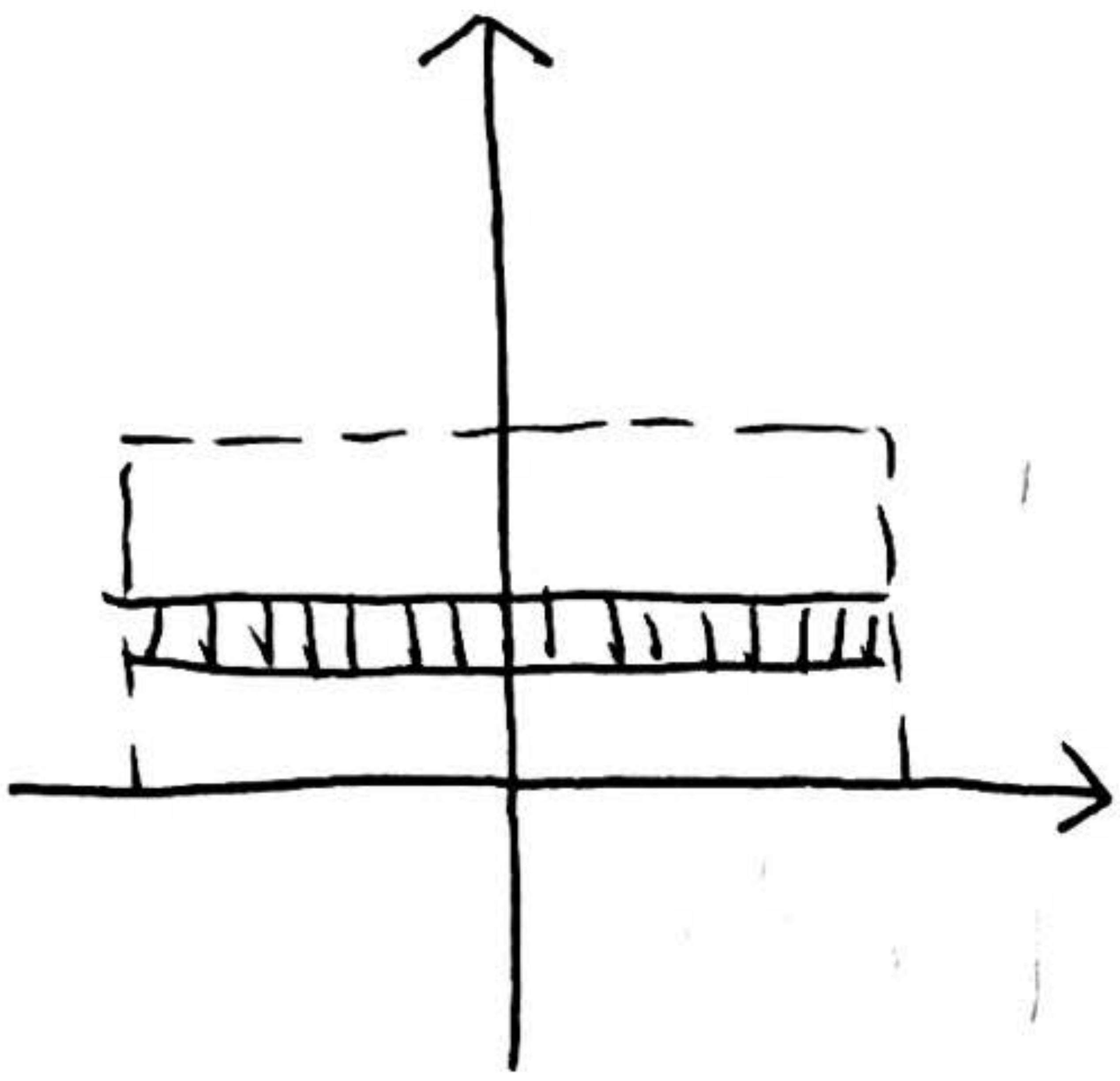
$$
\begin{aligned}
W_i &= F_i D_i \\
&= m_i a\,(6 - y_i) \\
&= \rho V_i g\,(6 - y_i) \\
&= \rho g\,(10 x_i \,\Delta y)(6 - y_i) \\
dW &= (62.5)(10(2y))(6-y)\,dy \\
&= (1250)\,(6y - y^2)\,dy
\end{aligned}
$$

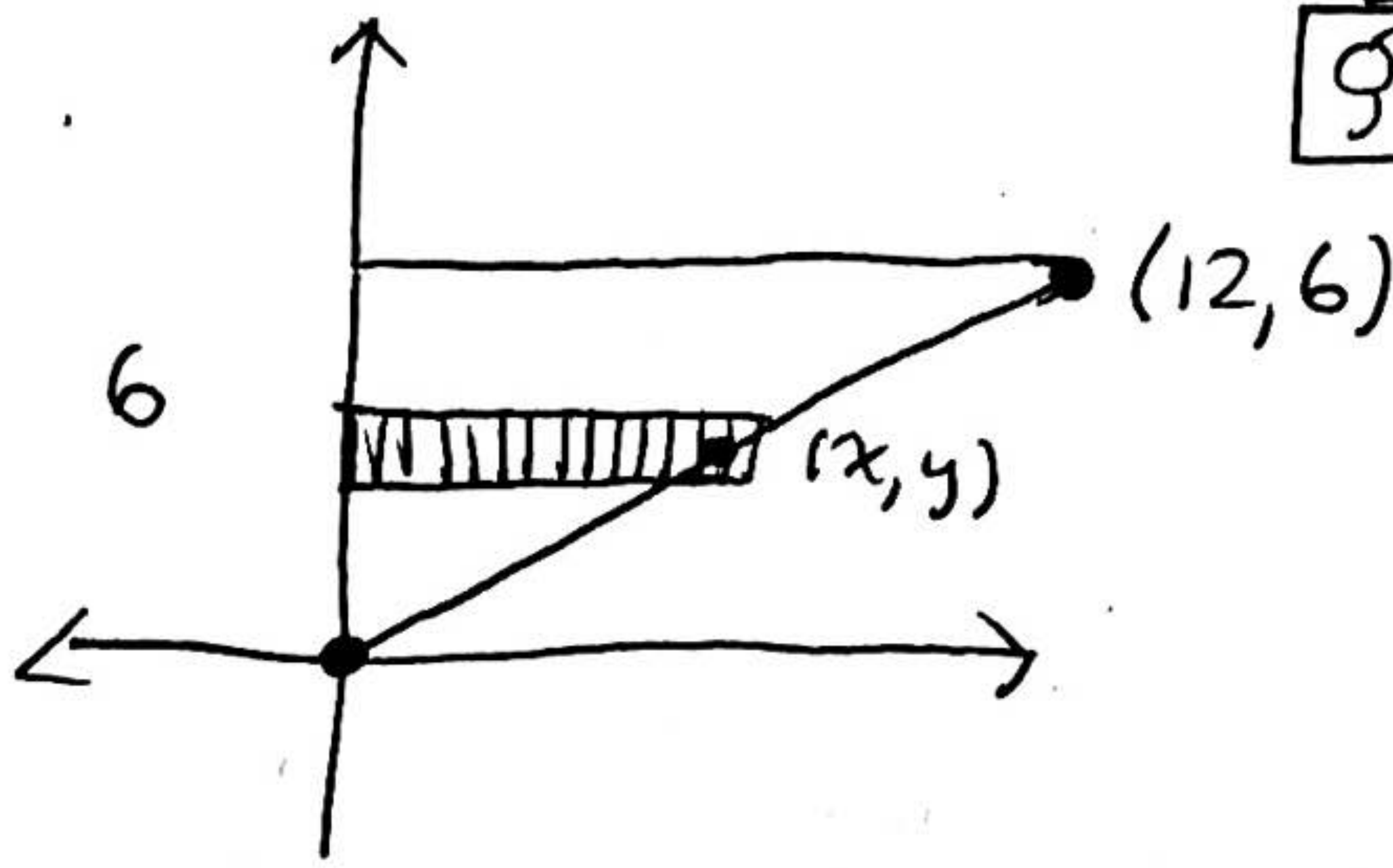
$$W = \int_0^6 20\rho g(6y - y^2)\,dy$$

$$
\begin{aligned}
W &= 1250\left[3y^2 - \tfrac{1}{3}y^3\right]_0^6 \\
&= 1250\left[3\cdot 6^2 - 2\cdot 6^2\right] \\
&= 1250(36) \\
&= \boxed{45{,}000 \text{ ft}\cdot\text{lbs}}
\end{aligned}
$$

E4

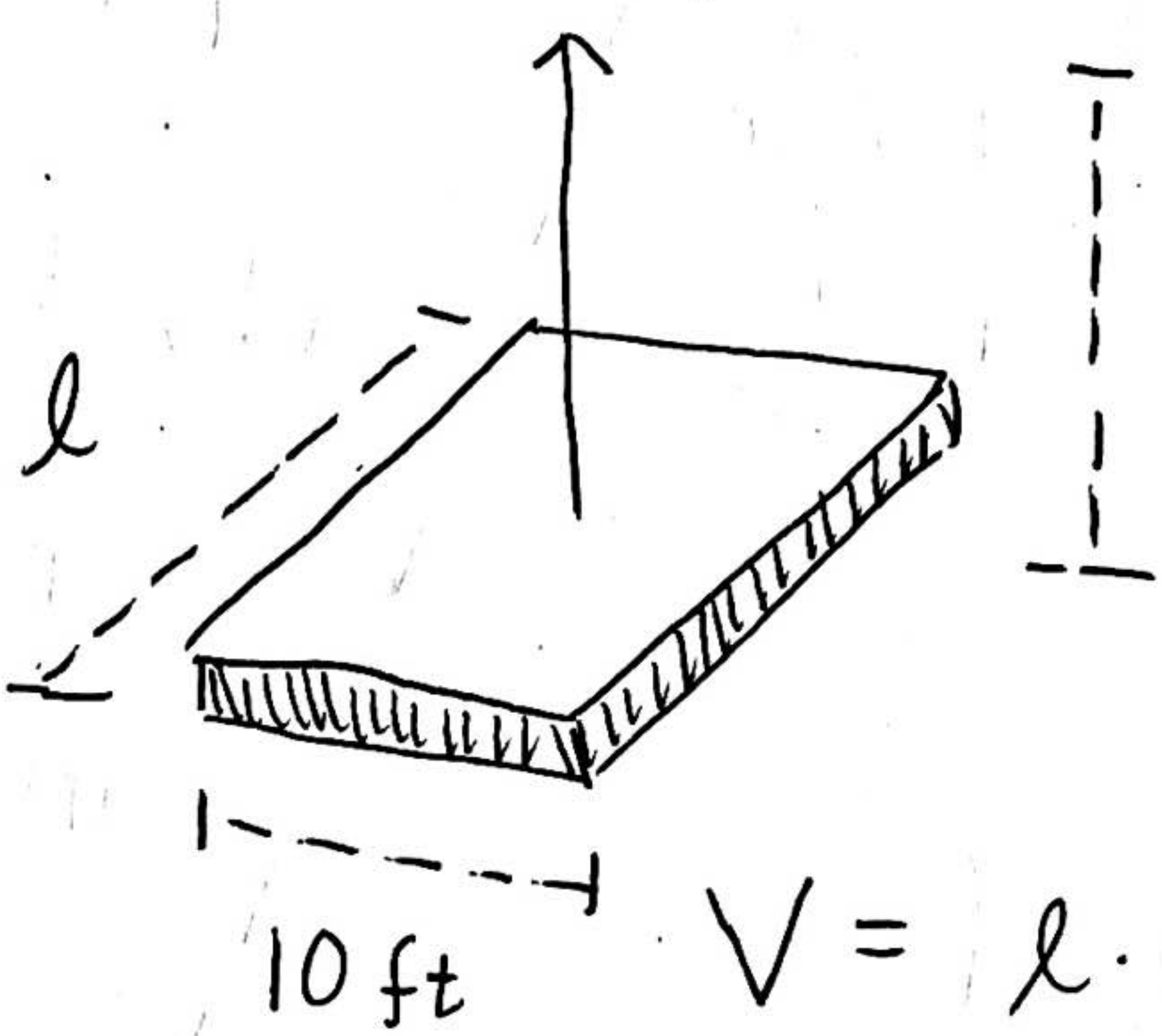
§6.4





$\ell = x$

$\ell = 2y$



$D = (6-y)$

$V = \ell \cdot w \cdot dy$

$dV = (2y)(10)\,dy$

$W = F \cdot D$

$= ma \cdot D$

$dW = \rho V g\,(6-y)$

$dW = \rho g(\ell \cdot w \cdot dy)(6-y)$

$= \rho g(2y)(10)(6-y)\,dy$

$W = \rho g \int_0^6 20y(6-y)\,dy$

Name: Answer Key

Worksheet: Sections 6.3
Volumes by Slicing

§6.2 OTHER EXAMPLE 2&3

1. Compute the volume of the solid whose base is a triangle with vertices at (0,0),(2,0),(0,2) and whose cross sections perpendicular to the base and parallel to the y-axis are semi-circles.

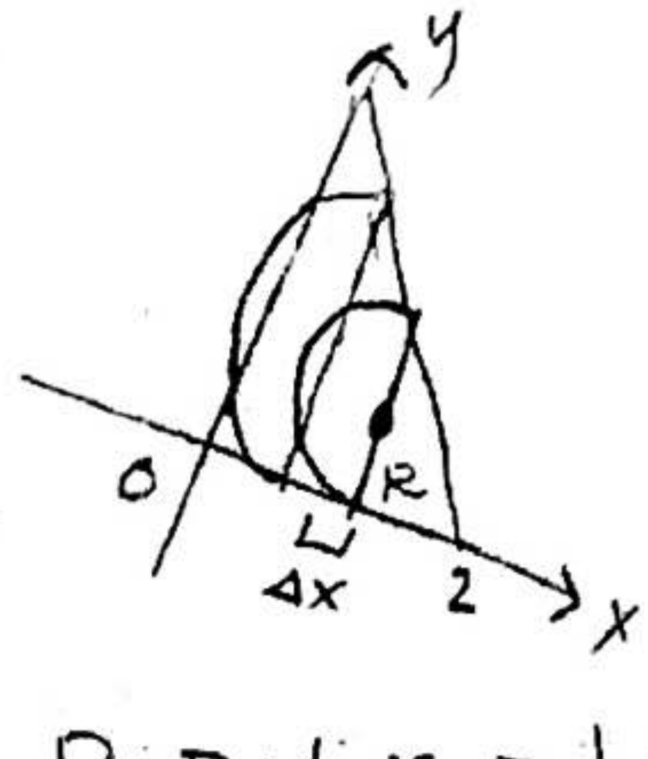
$A(x)$ = area of a semi-circle $= \frac{1}{2}\pi(R)^2$

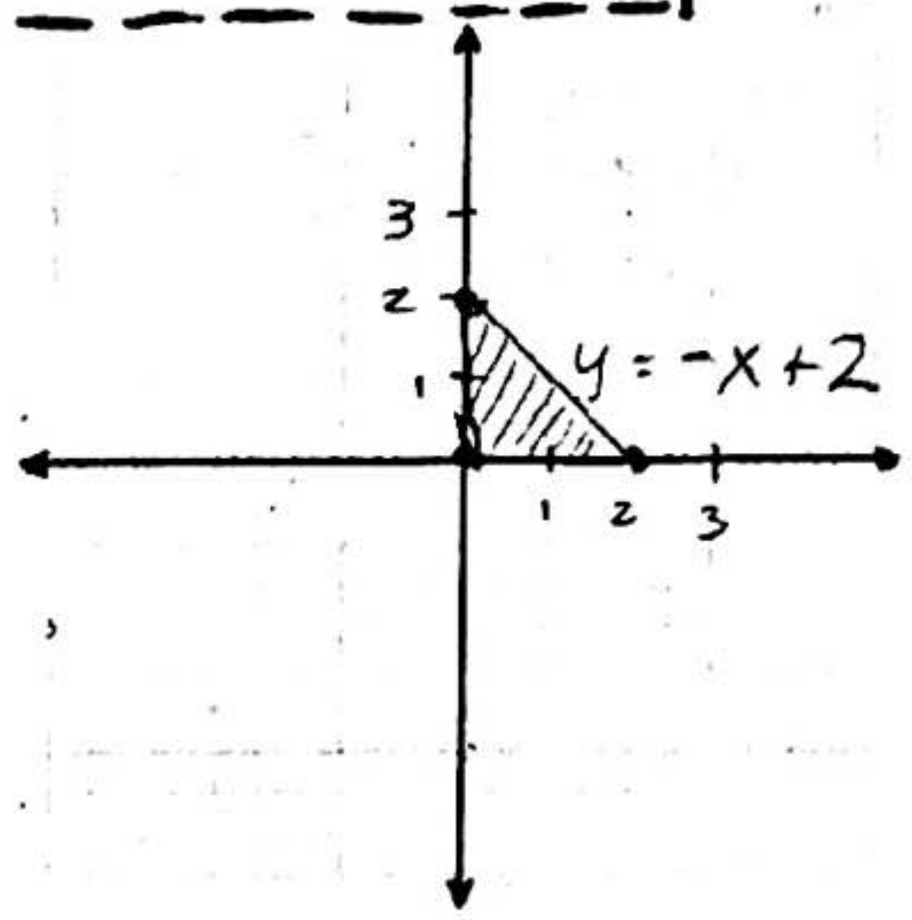
$= \frac{1}{2}\pi\left(\frac{1}{2}(2-x)\right)^2 = \frac{\pi}{8}(4-4x+x^2)$

Volume $= \int_0^2 \frac{\pi}{8}(4-4x+x^2)\,dx$

$= \frac{\pi}{8}\left(4x-2x^2+\frac{x^3}{3}\right)\Big|_0^2$

$= \frac{\pi}{8}\left(8-8+\frac{8}{3}\right)-0 = \boxed{\frac{\pi}{3}u^3}$





R = radius $= \frac{1}{2}y$

$= \frac{1}{2}(-x+2) = \frac{1}{2}(2-x)$

2. Compute the volume of the solid whose base is the region bounded by the parabola and line $y=x^2$ and $y=1$ where

a. The cross sections perpendicular to the base and parallel to the **y-axis** are squares.

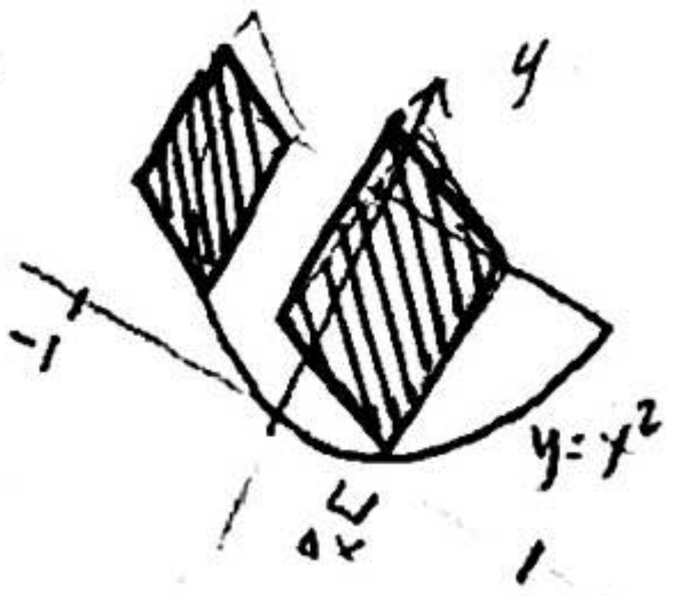
$A(x)$ = area of a square $= (\text{side})^2$

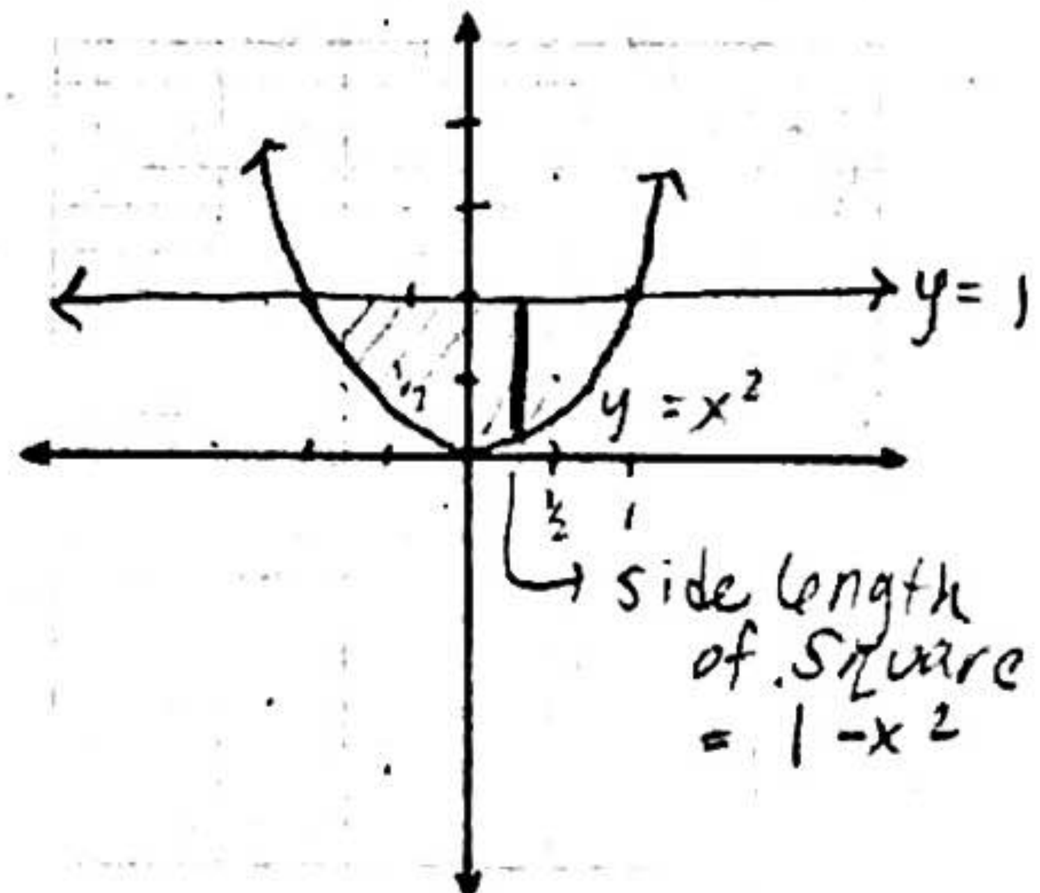
$= (1-x^2)^2 = (1-2x^2+x^4)$

Volume $= \int_{-1}^{1}(1-2x^2+x^4)\,dx$

$= 2\int_0^1 (1-2x^2+x^4)\,dx = 2\left[x-\frac{2}{3}x^3+\frac{x^5}{5}\right]_0^1 = 2\left[1-\frac{2}{3}+\frac{1}{5}\right] = \boxed{\frac{16}{15}u^3}$

15 − 10 + 3





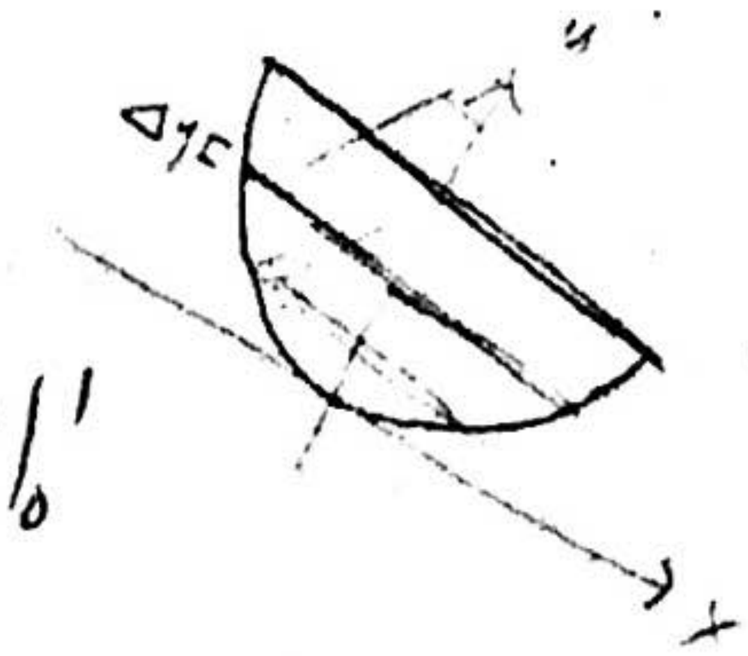
b. The cross sections perpendicular to the base and parallel to the **x-axis** are squares.

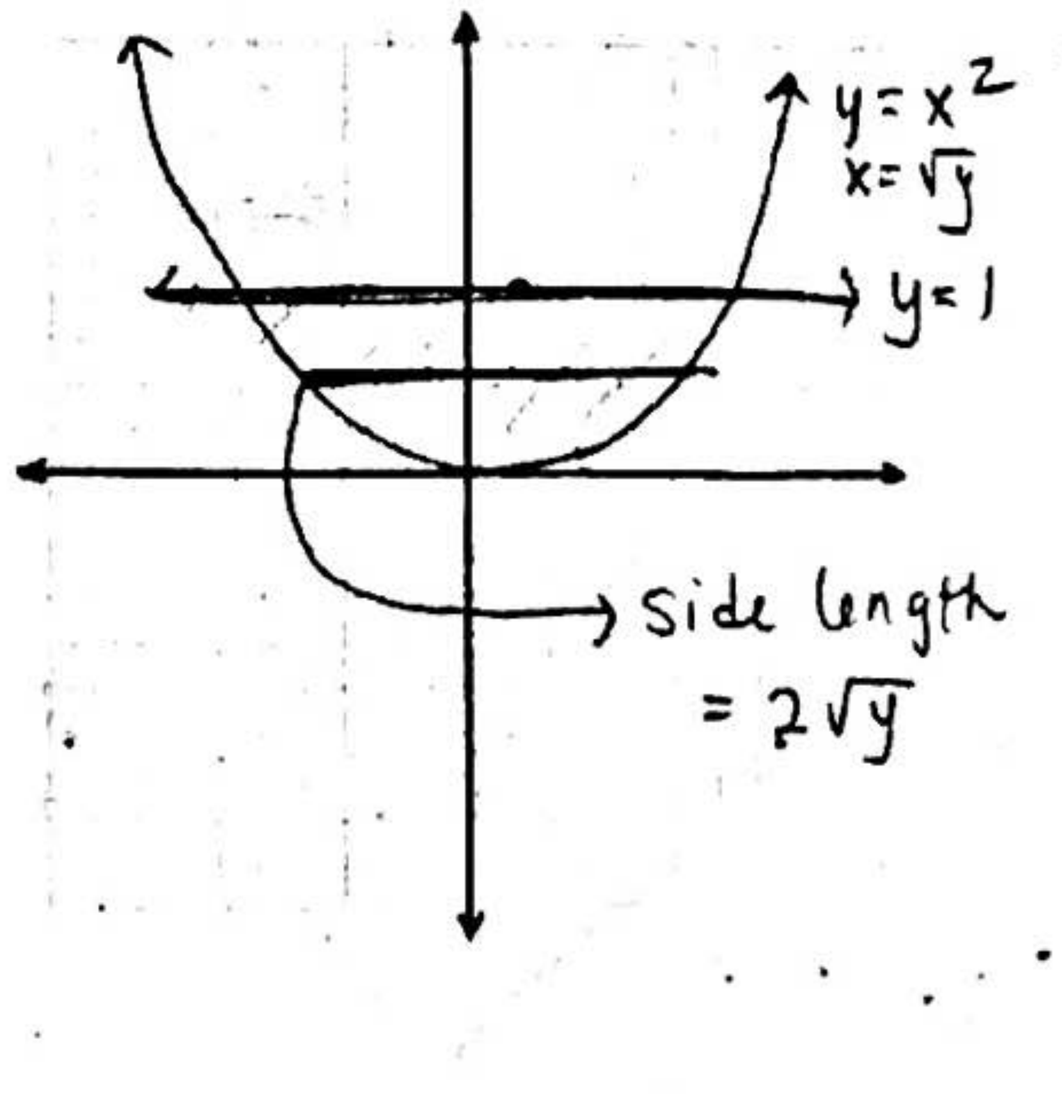
$A(y)$ = area of a square $= (\text{side})^2$

$= (2\sqrt{y})^2 = 4y$

Volume $= \int_0^1 4y\,dy = \frac{4y^2}{2}\Big|_0^1 = 2y^2\Big|_0^1$

$= \boxed{2u^3}$





THE BASE OF AN ELIPTICAL CURVE IS $9x^2+4y^2=36$. CROSS SECTIONS $\perp$ TO THE Y-AXIS ARE ISOSCELES RIGHT TRIANGLES W/ HYP. @ BASE. FIND VOLUME

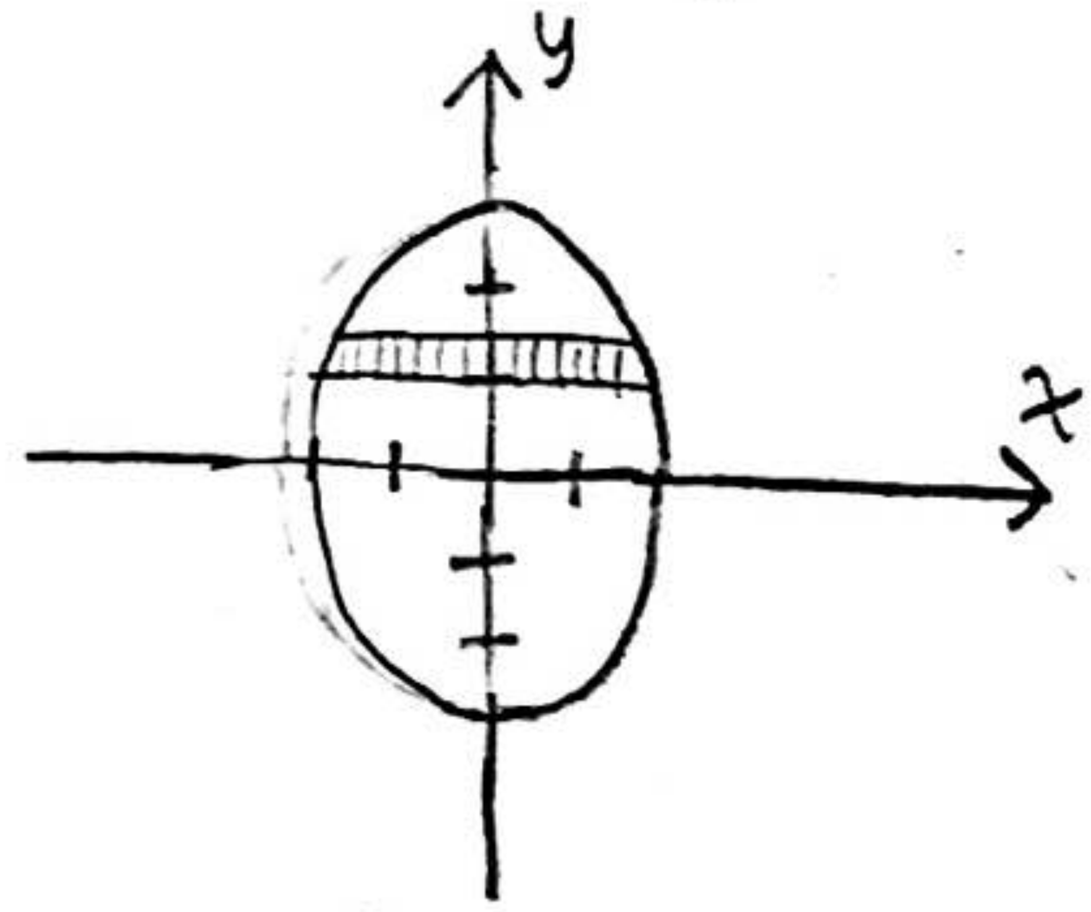
§6.2
EX B

VOLUME BY SLICING.

$$9x^2+4y^2=36$$

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

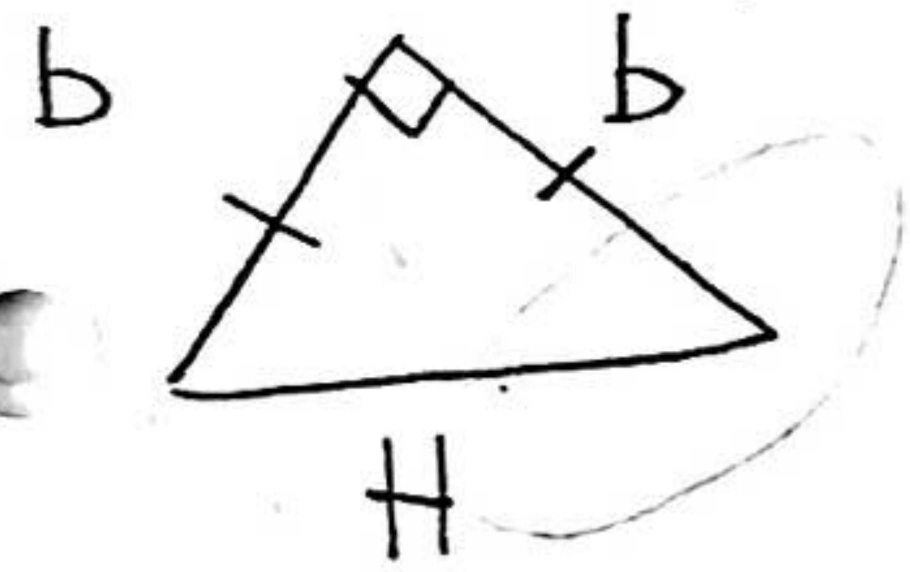
$$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$$

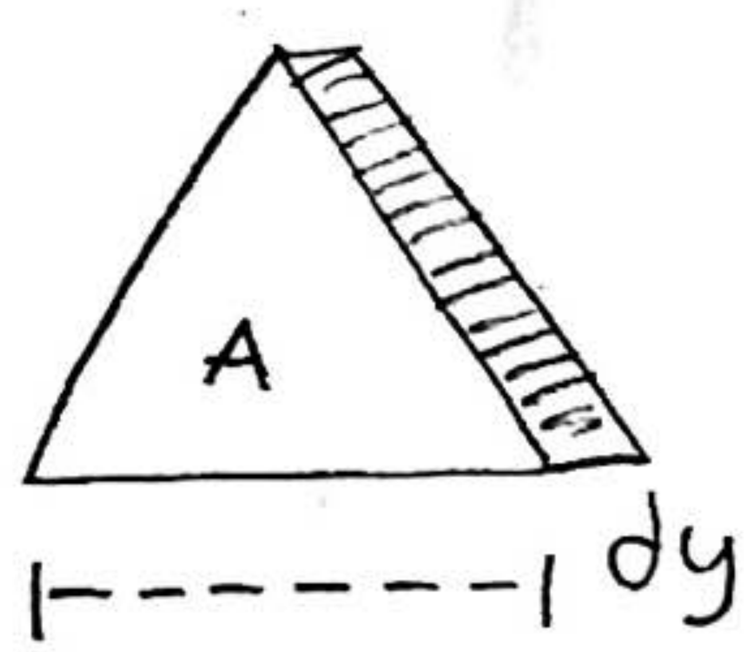


$$9x^2=36-4y^2$$

$$x^2=4-\frac{4y^2}{9}$$

$$x^2=4-\left(\frac{2y}{3}\right)^2$$





$$A=\frac{1}{2}bh$$

$$=\frac{1}{2}bb$$

$$\boxed{A=\frac{1}{2}b^2}$$

$$b^2+b^2=H^2$$

$$2b^2=H^2$$

$$\boxed{b^2=\frac{1}{2}H^2}$$

$$A=\frac{1}{2}\left(\frac{1}{2}H^2\right)$$

$$=\frac{1}{4}H^2$$

$$H=2x$$

$$H^2=4x^2$$

$$\frac{1}{4}H^2=x^2$$

$$A=x^2$$

$$A=4-\left(\frac{2y}{3}\right)^2$$

$$V=A\,dy$$

$$V=4-\left(\frac{2y}{3}\right)^2$$

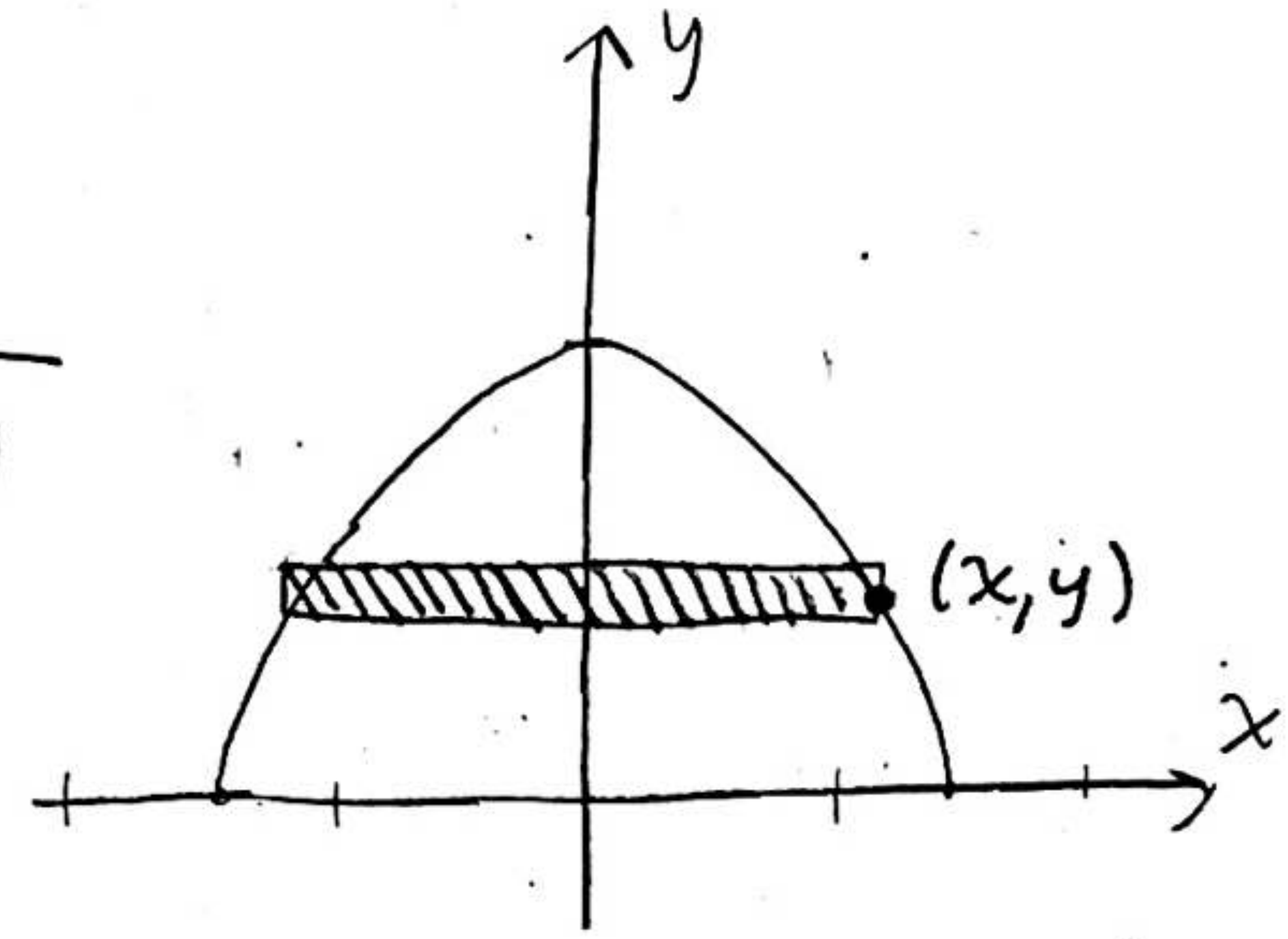
$$-3\le y\le 3$$

$$\int_{-3}^{3}\left(4-\frac{4y^2}{9}\right)dy=\int_0^3(2)\left[4-\frac{4}{9}y^2\right]dy$$

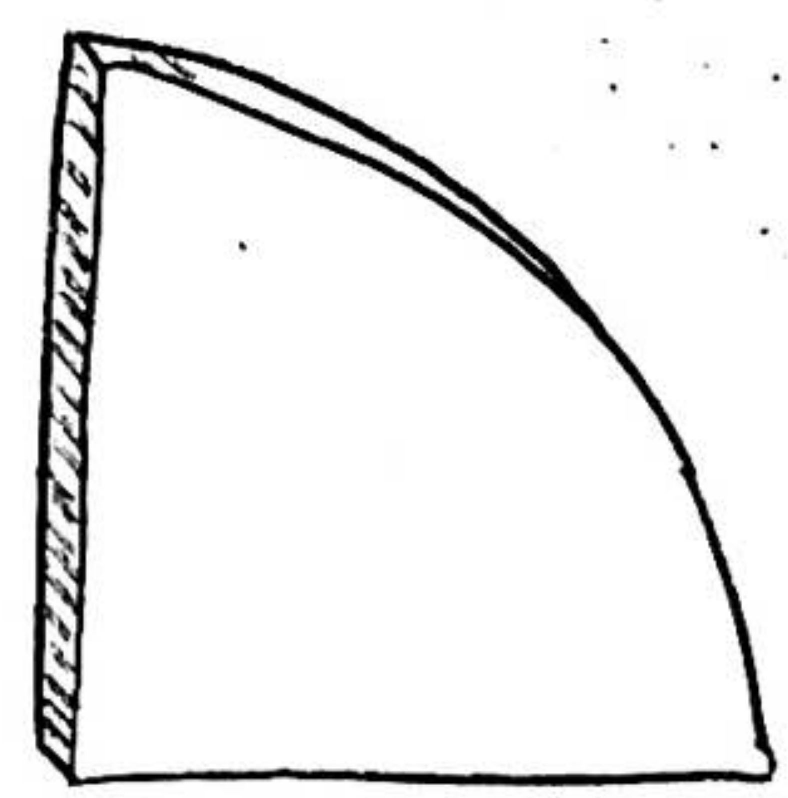
$$=\left(4y-\frac{4}{27}y^3\right)(2)\Big|_0^3=\left[4(3)-\frac{4}{27}27\right](2)=16u^2$$

Find the volume of the solid S is the region enclosed by $y = 2 - x^2$ and the x-axis. Cross-sections perpendicular to the y-axis are quarter circles. Find the volume.

$$y = 2 - x^2$$
$$x^2 = 2 - y$$
$$x = \pm\sqrt{2-y}$$



$$\int_0^2 \pi(2-y)\,dy$$
$$= \pi \int_0^2 2 - y \, dy$$
$$= \pi \left[2y - \frac{1}{2}y^2\right]_0^2$$
$$= \pi[4-2]$$
$$= 2\pi$$



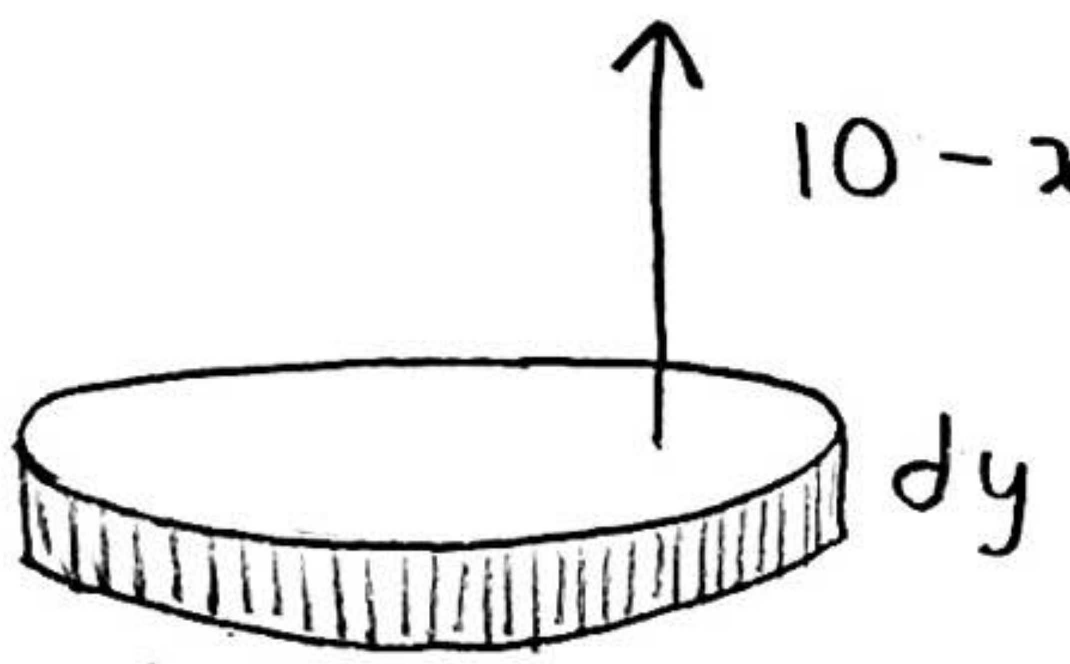
$$dV = A\,dy$$
$$dV = \frac{1}{4}\pi R^2\,dy$$
$$dV = \frac{1}{4}\pi(2x)^2\,dy$$
$$dV = \frac{4}{4}\pi(\sqrt{2-y})^2\,dy$$
$$dv = \pi(2-y)\,dy$$

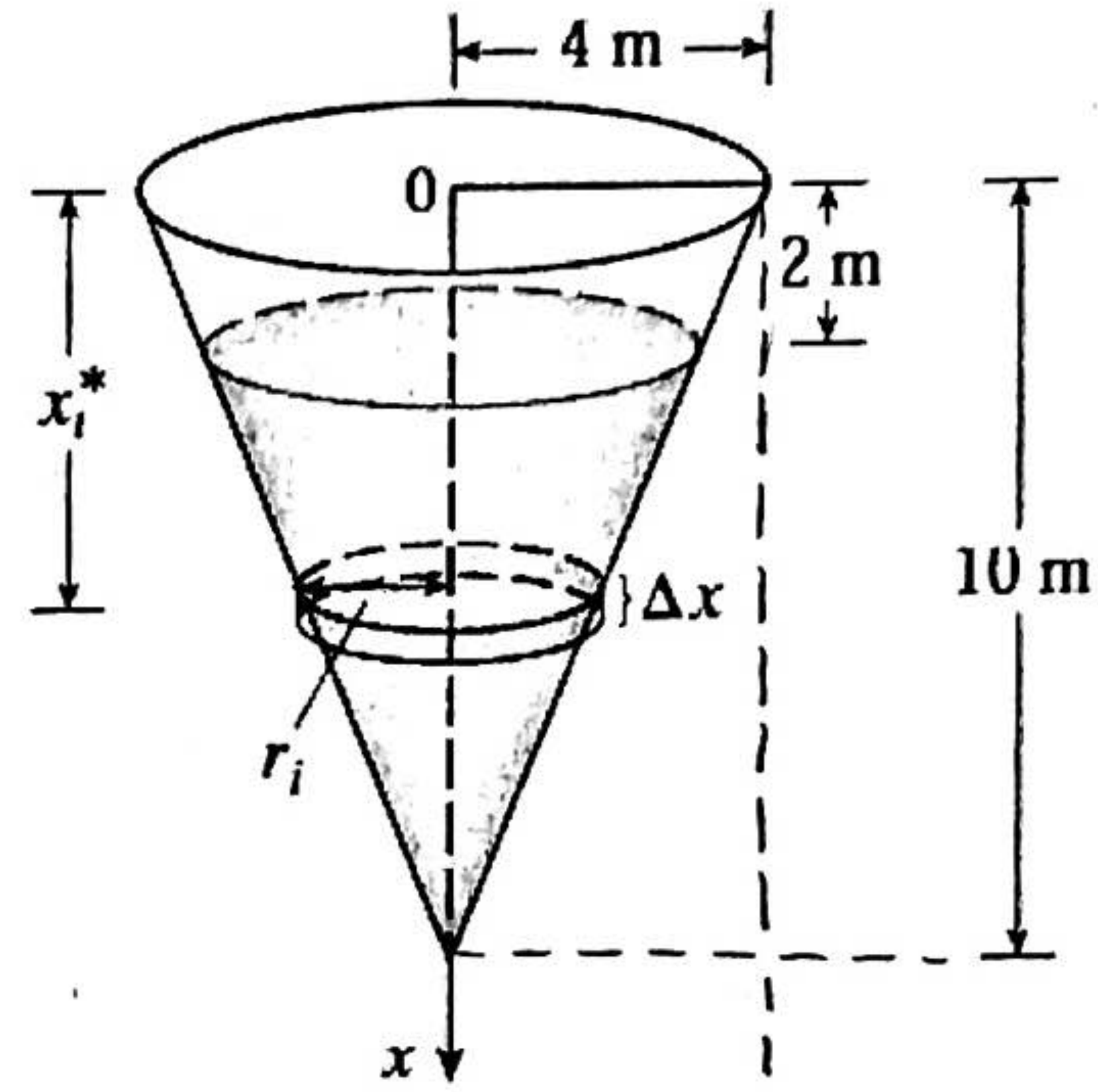
MUST LABEL CLEARLY WHERE YOUR AXIS RESIDES.

Math 31 | §6.4 Work

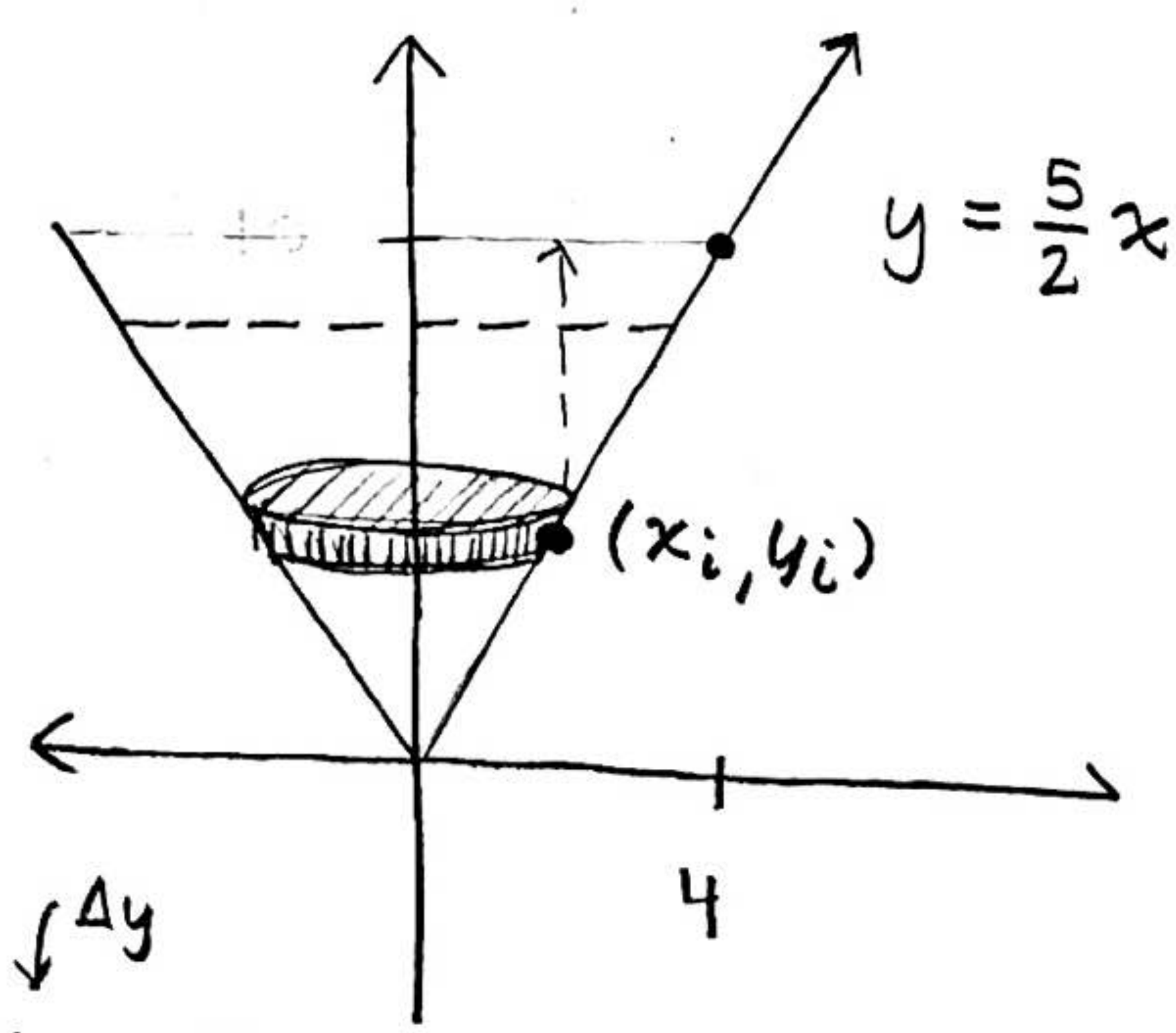
Example

A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water to a height of 8 m. Find the work required to empty the tank by pumping all of the water to the top of the tank.





$$W_i = F_i d_i$$
$$= m_i a d_i$$
$$= \rho V_i a d_i$$
$$= \rho g (\pi x_i^2)(10 - x_i)\, dy$$
$$= \rho g \left(\pi \tfrac{4}{25} y_i\right)\left(10 - \tfrac{2}{5} y_i\right) dy$$
$$dW = 9800\pi\left(\tfrac{4}{25} y\right)\left(10 - \tfrac{2}{5} y\right) dy$$



Δy

$$\approx \sum_{k=1}^{n} \rho g \left(\tfrac{4\pi}{25} y_k\right)\left(10 - \tfrac{2}{5} y_k\right)\left[\frac{\frac{16}{5} - 0}{n}\right]$$

$y_i = \frac{5}{2} x_i$

$x_i = \frac{2}{5} y_i$

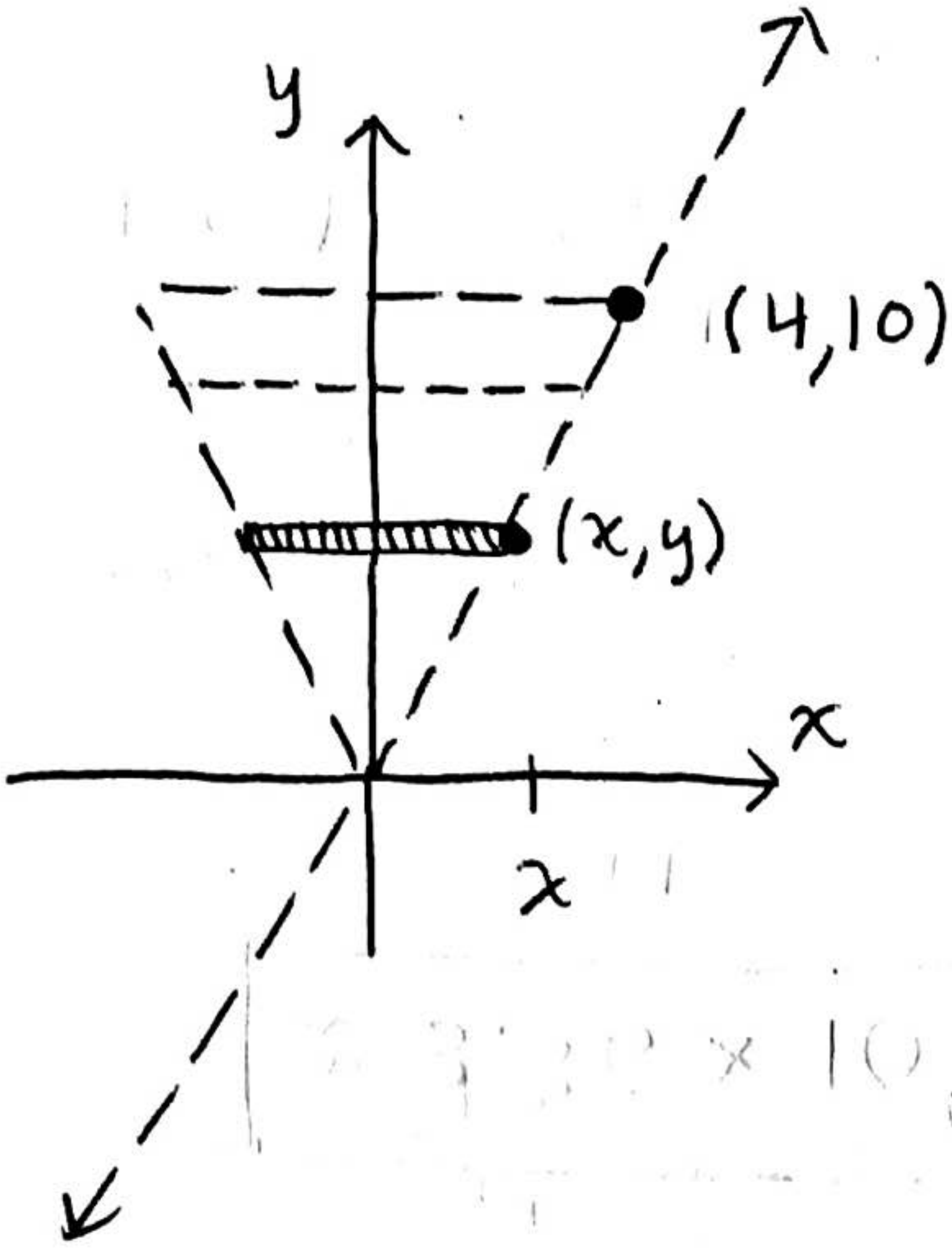
$$\int_0^{\frac{16}{5}} \rho g \left(\frac{4\pi}{25}\right)(y)\left(10 - \frac{2}{5} y\right) dy$$

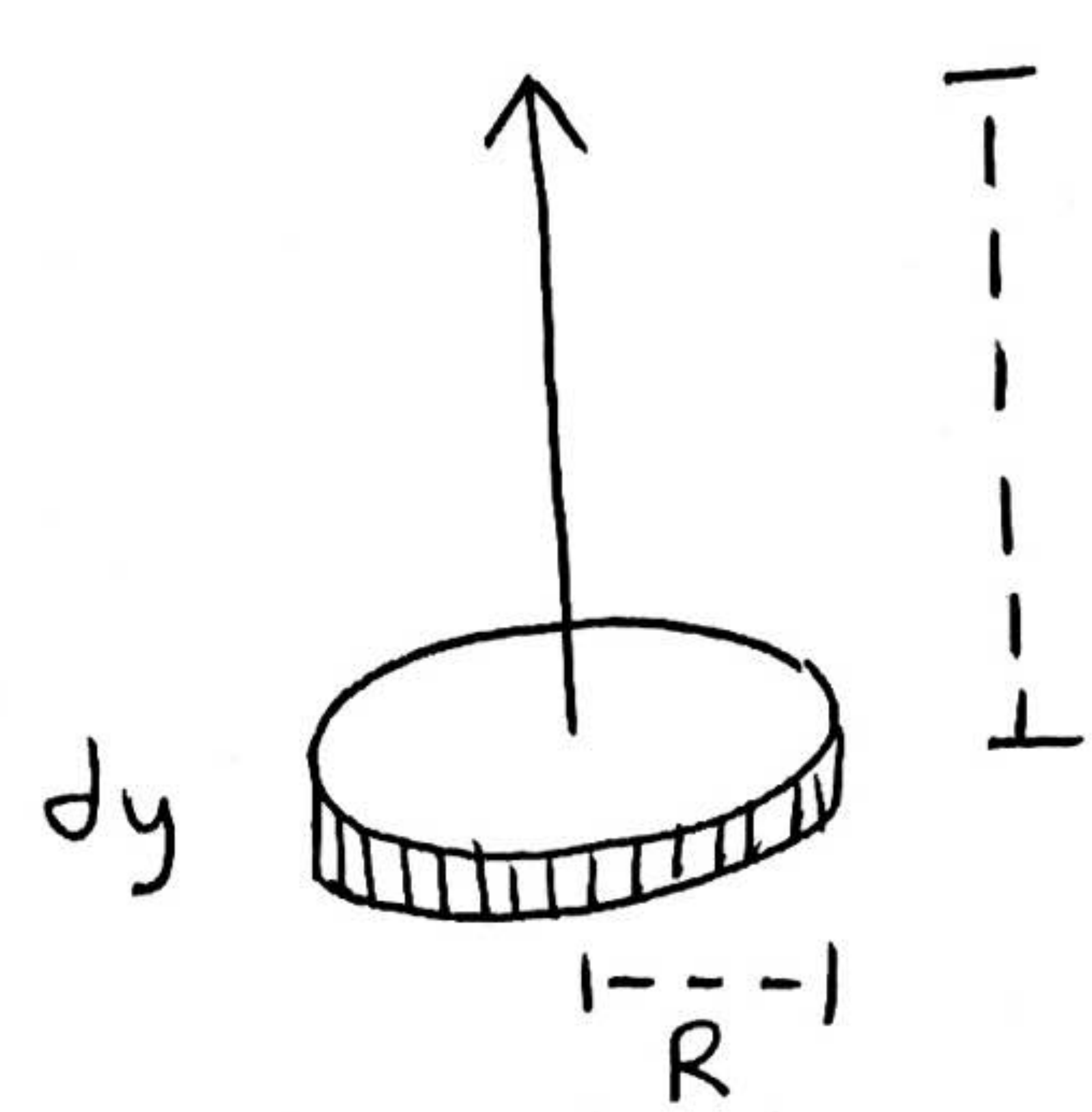
$$= \frac{9800}{25} \cdot 4\pi \int_0^{\frac{16}{5}} 10y - \frac{2}{5} y^2 \, dy$$

$\boxed{\S 6.4}$

E3

§6.4





$$W = F \cdot D$$
$$= ma \cdot D$$
$$= \rho V g \cdot D$$

$$y = \frac{10}{4}x$$
$$y = \frac{5}{2}x$$

$$dW = \rho g (A\,dy)\,D$$
$$= \rho g (D) A\,dy$$
$$= \rho g D (\pi R^2)\,dy$$

$$D = 10 - y$$
$$R = x$$
$$= \frac{2}{5}y$$
$$\rho = 10^3$$
$$g = 9.8$$

$$dW = \rho g (10-y)\pi\left(\tfrac{2}{5}y\right)^2 dy$$

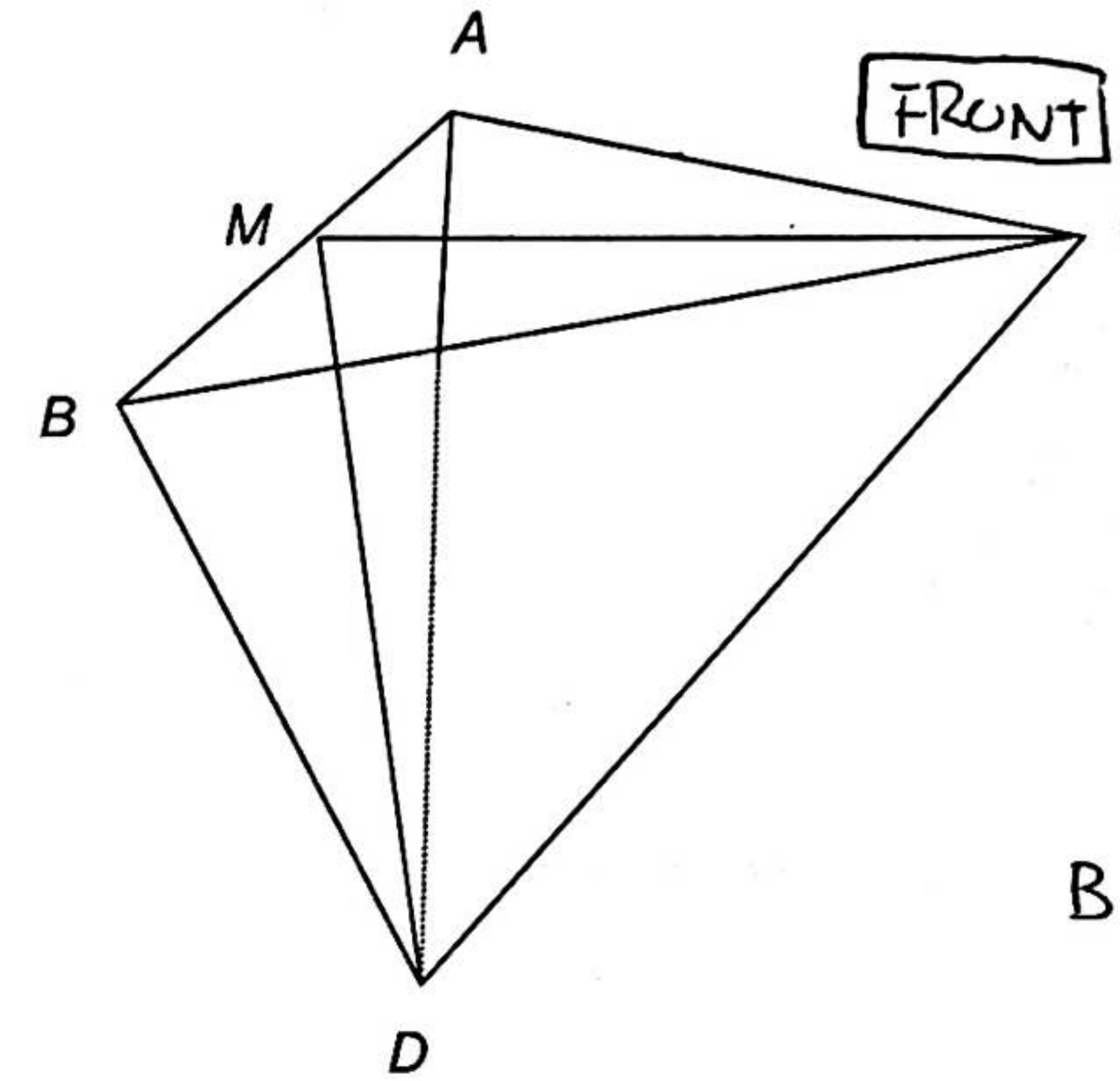
$$\int_0^8 dW = \int_0^8 \rho g \pi (10-y)\left(\tfrac{2}{5}y\right)^2 dy$$
$$= 9800\pi \int_0^8 (10-y)\left(\tfrac{2}{5}y\right)^2 dy$$
$$= 9800\pi\left(\frac{8192}{75}\right) = \frac{3211264\pi}{3}$$

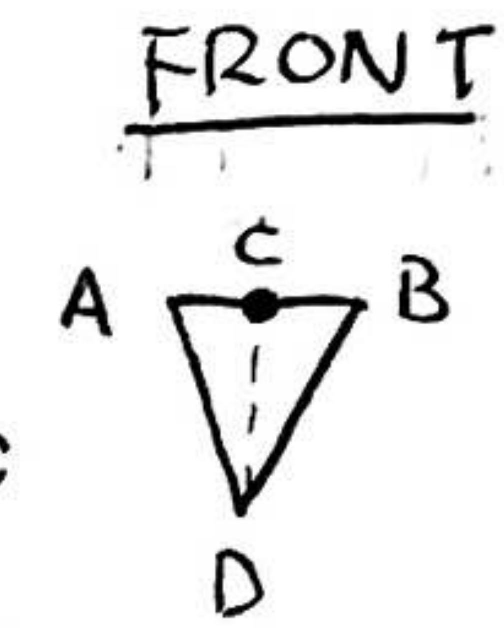
CHARLES ALBRIGHT

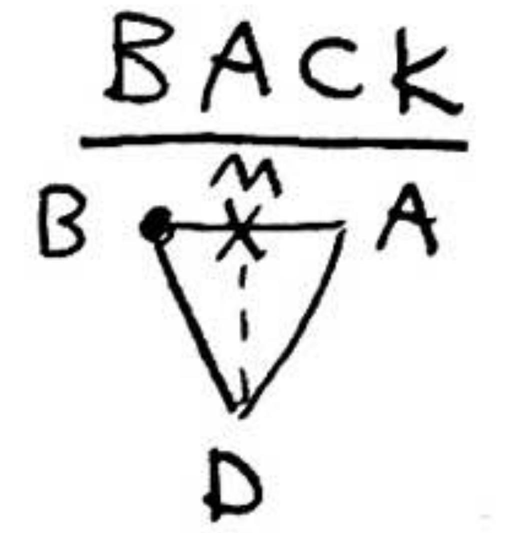
Provide a presentation that is both clear and organized. Show all of your work, simplify results, and give exact values only.

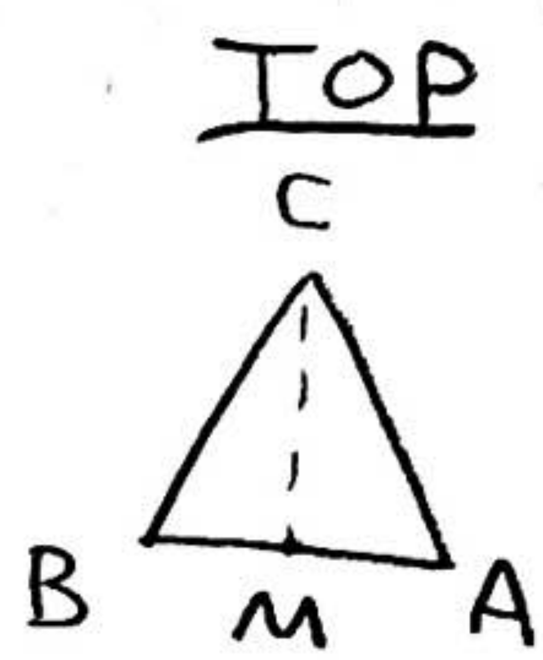
A tank is in the shape of a pyramid as depicted in the following picture. Take into consideration the following qualities that this pyramid possesses: $\overline{CM} \perp \overline{DM}$, $\overline{CM} \perp \overline{AB}$, $\overline{DM} \perp \overline{AB}$, $\overline{AD} \cong \overline{BD}$, and $\overline{AC} \cong \overline{BC} \cong \overline{AB}$. Also note that $DM = 30$ m, $CM = 40$ m, and the triangle ΔABC is parallel to the horizontal.

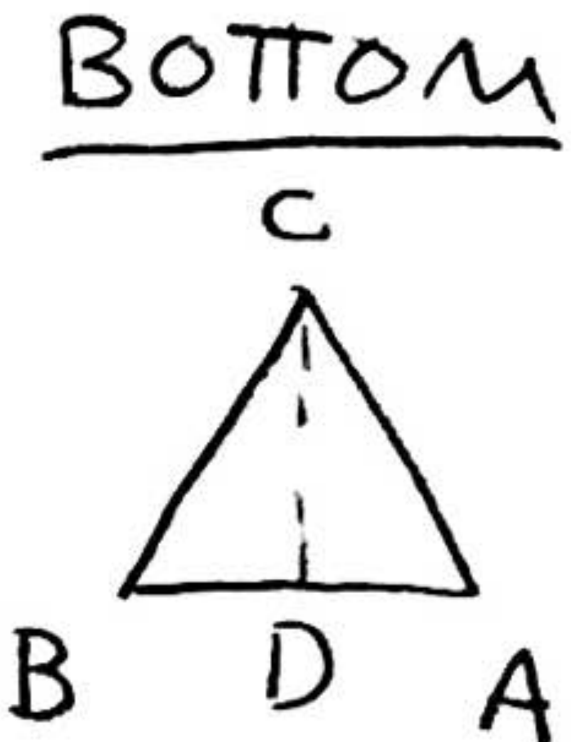
BACK

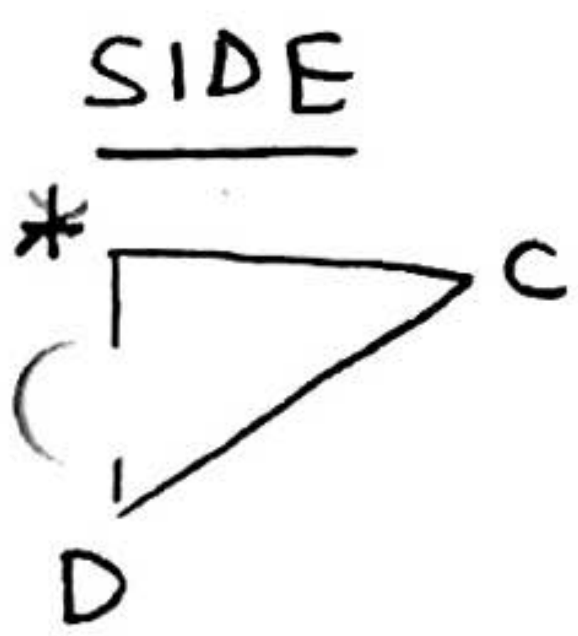








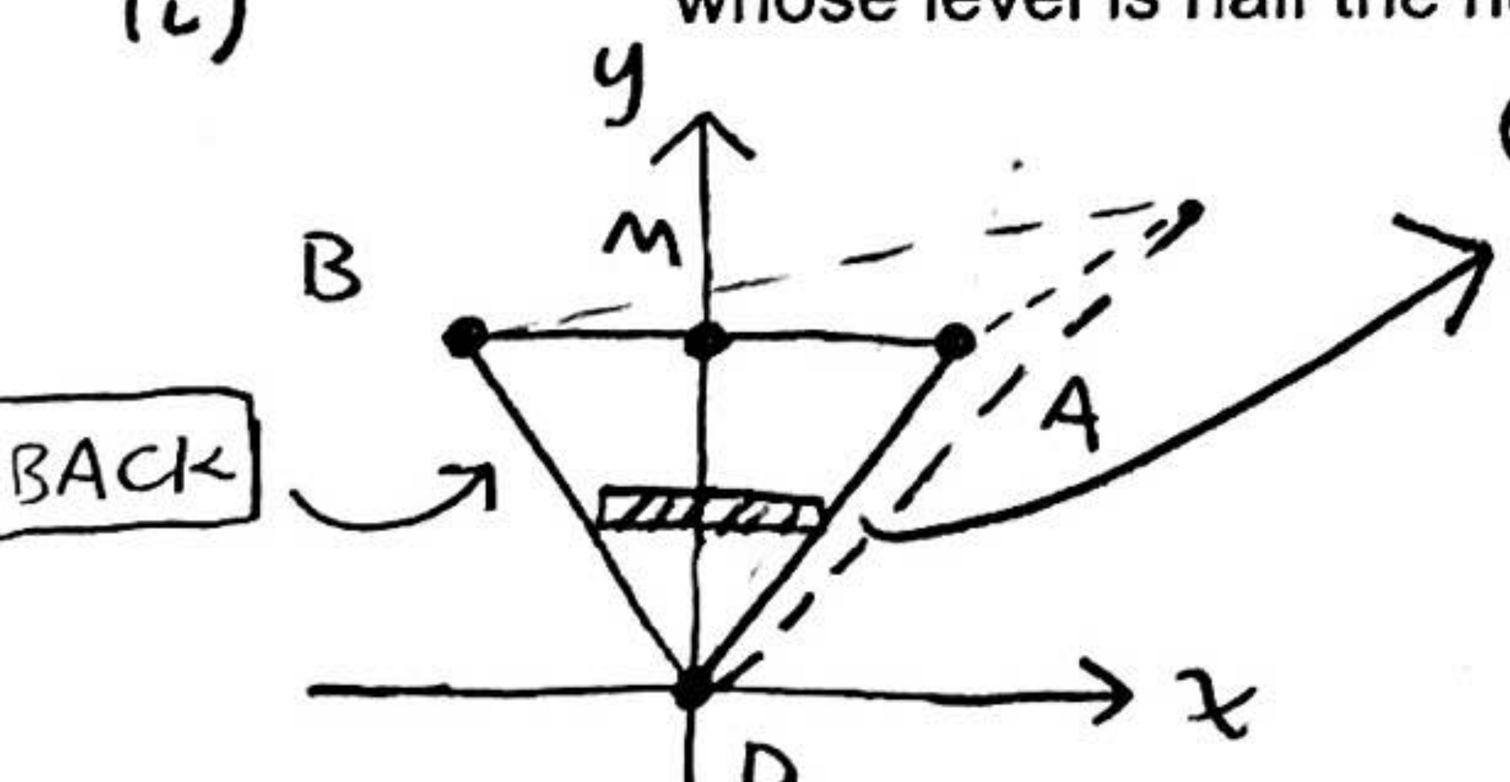




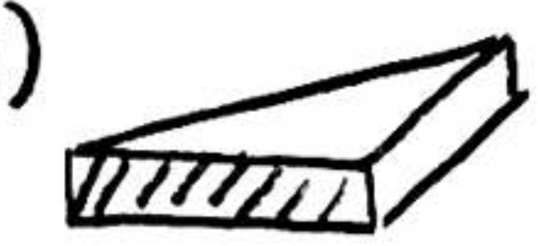
i) If this tank is full of water, then how much work is required to pump the water to the top, which is open, so that the water flows out?

ii) What if the top is closed and there is a spout at the top out from the water flows and the spout is 5 m tall? Assume that the tank is not full, but filled with water whose level is half the height of this tank.

(i)

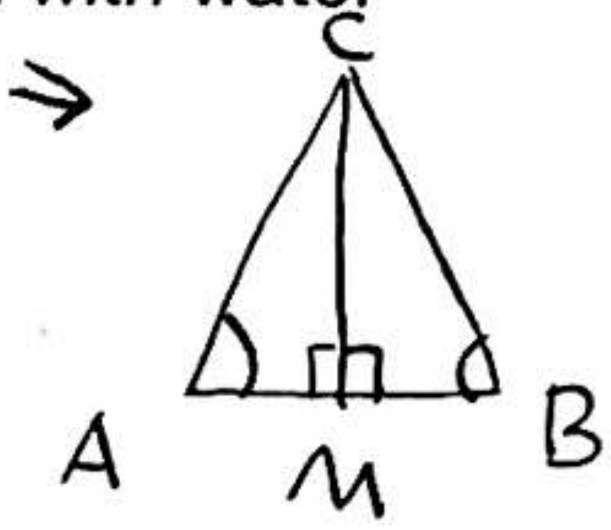


(1)



VOLUME (V)

$V = \frac{\sqrt{3}}{4} s^2 dy$



BY SPECIAL TRIANGLES.

THUS,

$CM = \frac{\sqrt{3}}{2} AB$

$40 = \frac{\sqrt{3}}{2} AB$

$AB = \frac{80\sqrt{3}}{3}$

FACE OF SLICE Δ SIMILAR TO ΔABC.

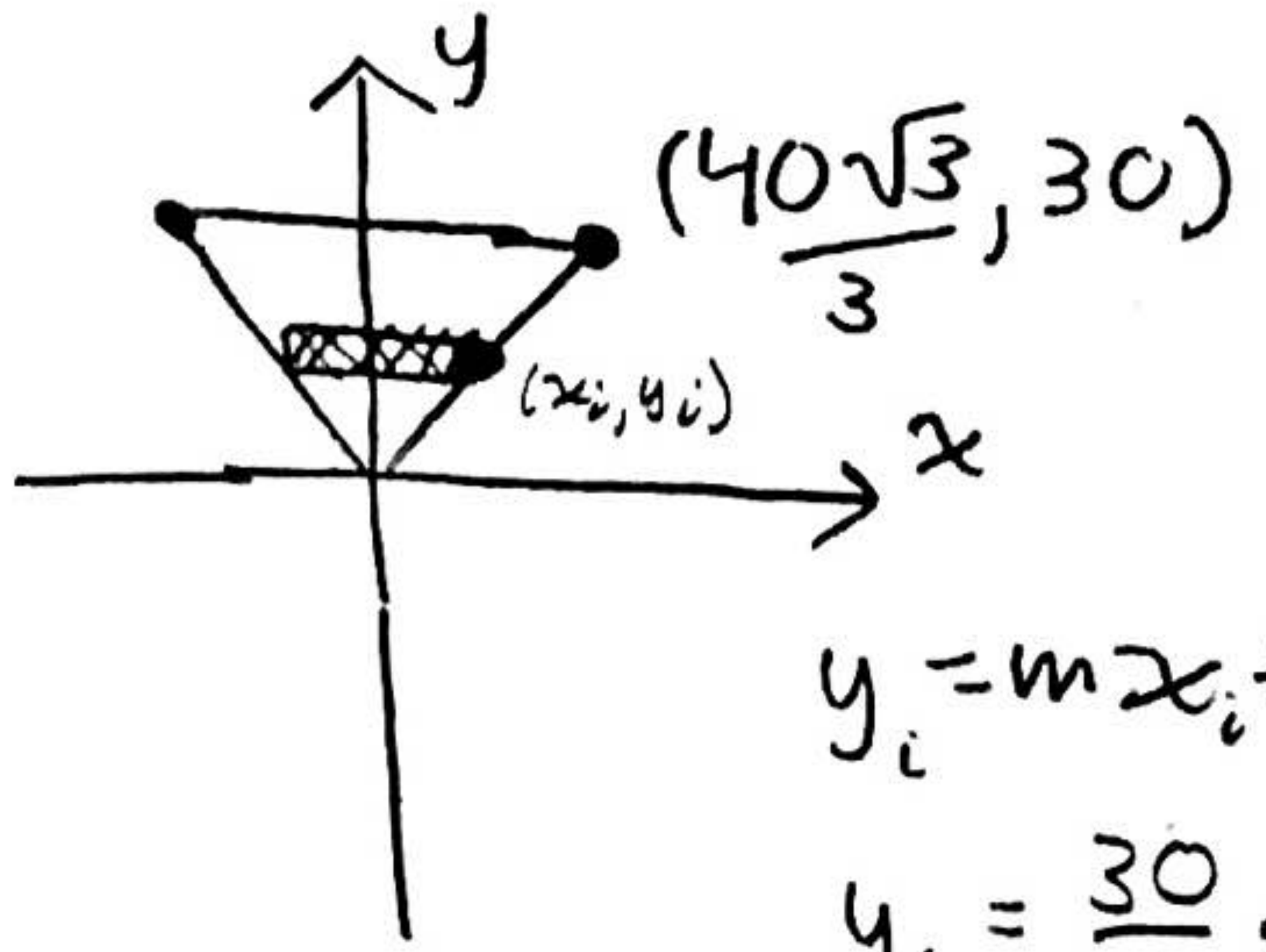
(2) ALSO,

$CM = \frac{\sqrt{3}}{2} AB$

$D = (0,0)$

$A =$

(OVER)



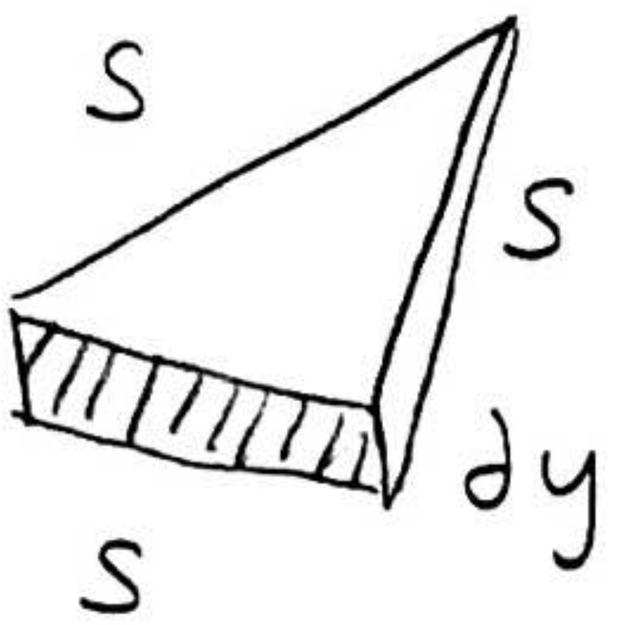
$y_i = m x_i + b$

$y_i = \frac{30}{1} \cdot \frac{3}{40\sqrt{3}} x_i + 0$

$y_i = \frac{90}{40\sqrt{3}} x_i$

$y_i = \frac{9}{4\sqrt{3}} x_i$

$x_i = \frac{4\sqrt{3}}{9} y_i$



$V_i = A_i dy$

$= (2x_i) dy$

$= \frac{8\sqrt{3}}{9} y_i dy$

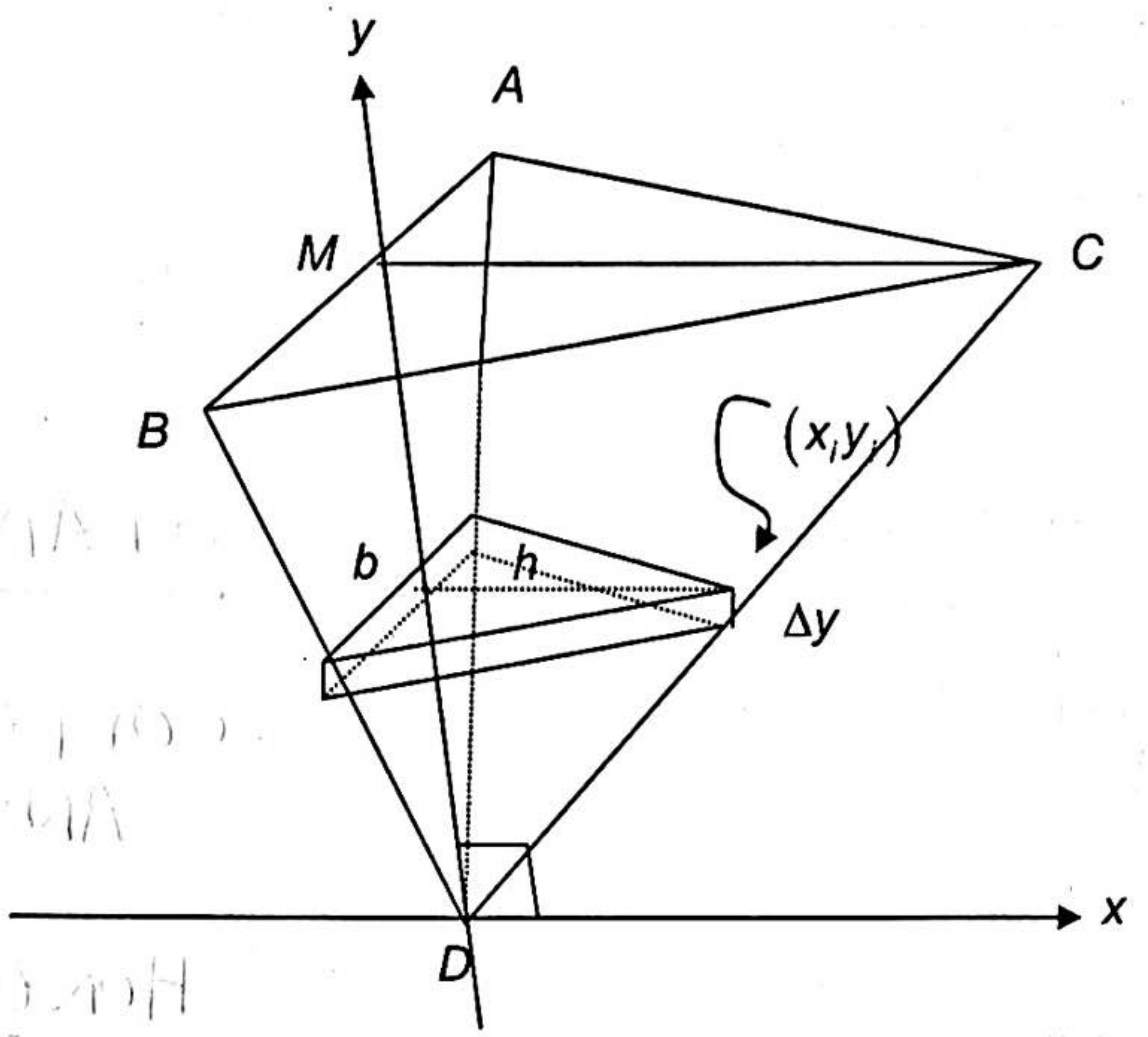
$W_i = (30 - y_i)\, 9800 \left(\frac{8\sqrt{3}}{9}\right) y_i dy$

$W_i = \frac{9800(8\sqrt{3})}{9} y_i (30 - y_i) dy$

$= \int_0^{30} \frac{9800(8\sqrt{3})}{9} y(30-y) dy$

$= \frac{78{,}400\sqrt{3}}{9} \int_0^{30} 30y - y^2 \, dy$

TAKE HOME QUIZ SOLNS



- $v_i = A_{\text{triangle}} \cdot \text{thickness}$

$$= \frac{1}{2} \cdot \text{base} \cdot \text{height} \cdot \Delta y$$

$$= \frac{1}{2} \cdot b \cdot h \cdot \Delta y \text{ where by similar triangles, } \frac{2b}{40\sqrt{3}} = \frac{h}{40}, \text{ or } b = \frac{\sqrt{3}}{2}h$$

$$= \frac{\sqrt{3}}{4} h^2 \Delta y \text{ where, again, by similar triangles, } \frac{h}{40} = \frac{y}{30}, \text{ or } h = \frac{4}{3}y$$

$$= \frac{\sqrt{3}}{4}\left(\frac{4}{3}y_i\right)^2 \Delta y$$

$$= \frac{4\sqrt{3}}{9} y_i^2 \Delta y$$

- $f_i = \rho \cdot g \cdot v_i$

$$= 1000(9.8)\frac{4\sqrt{3}}{9} y_i^2 \Delta y$$

$$= \frac{39{,}200\sqrt{3}}{9} y_i^2 \Delta y$$

- $w_i = f_i d_i$

$$= \frac{39{,}200\sqrt{3}}{9} y_i^2 \Delta y \cdot (30 - y_i)$$

*

$$= \frac{39{,}200\sqrt{3}}{9} (30 - y_i) y_i^2 \Delta y$$

$$= \frac{39{,}200\sqrt{3}}{9} \left(30y_i^2 - y_i^3\right) \Delta y$$

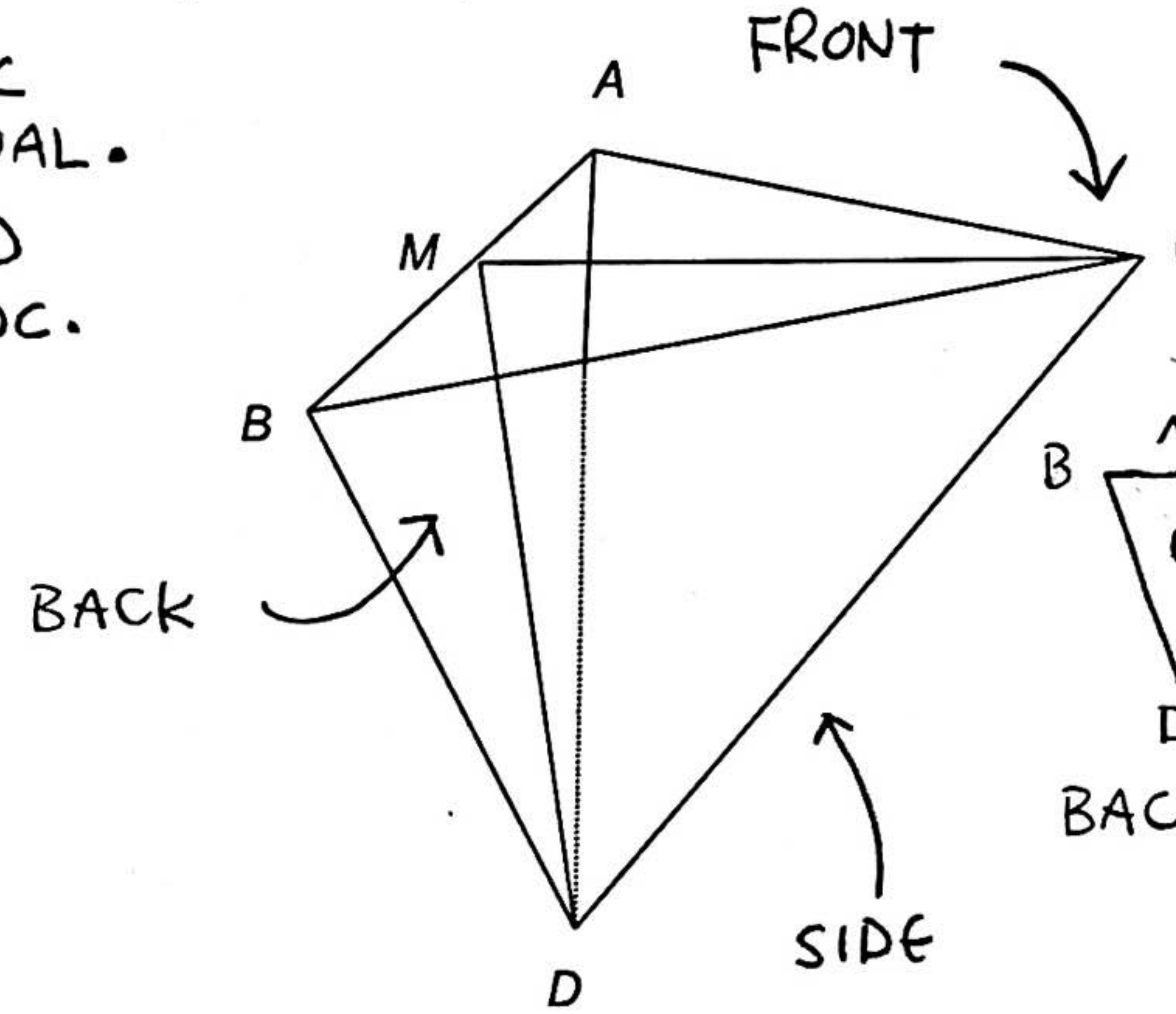
Math 31 **Quiz 8** **September 23, 2016**

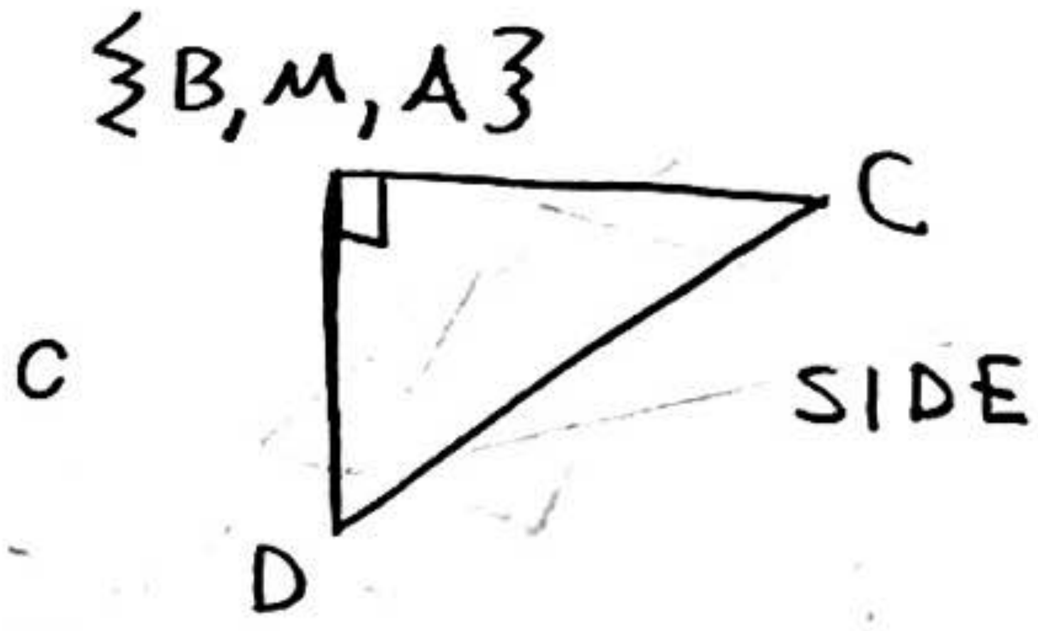
Provide a presentation that is both clear and organized. Show all of your work, simplify results, and give exact values only.

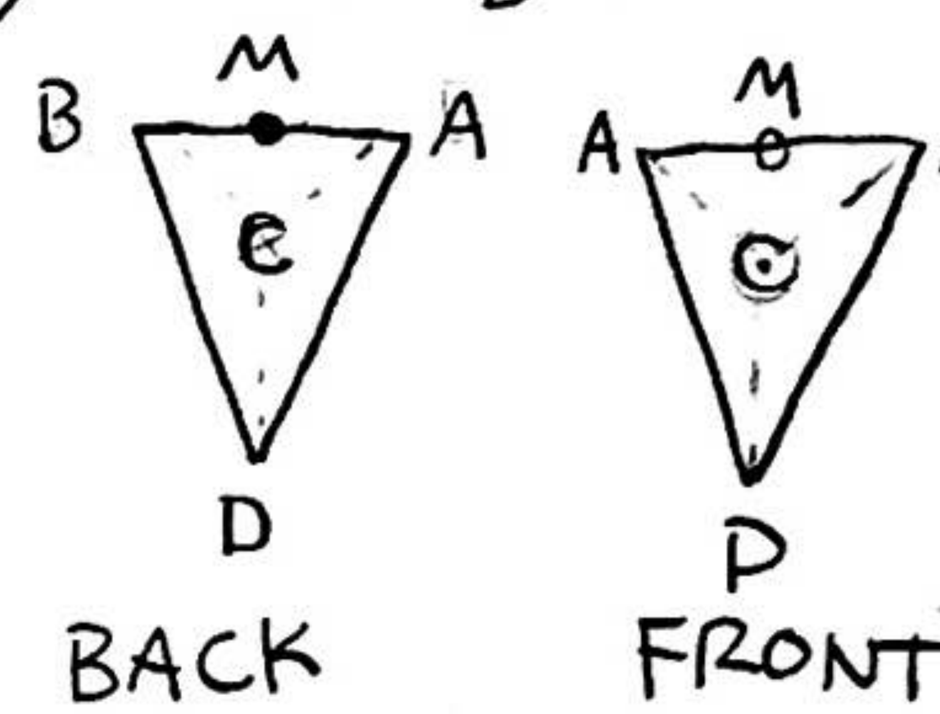
A tank is in the shape of a pyramid as depicted in the following picture. Take into consideration the following qualities that this pyramid possesses: $\overline{CM} \perp \overline{DM}$, $\overline{CM} \perp \overline{AB}$, $\overline{DM} \perp \overline{AB}$, $\overline{AD} \cong \overline{BD}$, and $\overline{AC} \cong \overline{BC} \cong \overline{AB}$. Also note that $DM = 30$ m, $CM = 40$ m, and the triangle ΔABC is parallel to the horizontal.

Δ ABC IS EQUAL.

Δ ABD IS ISOC.

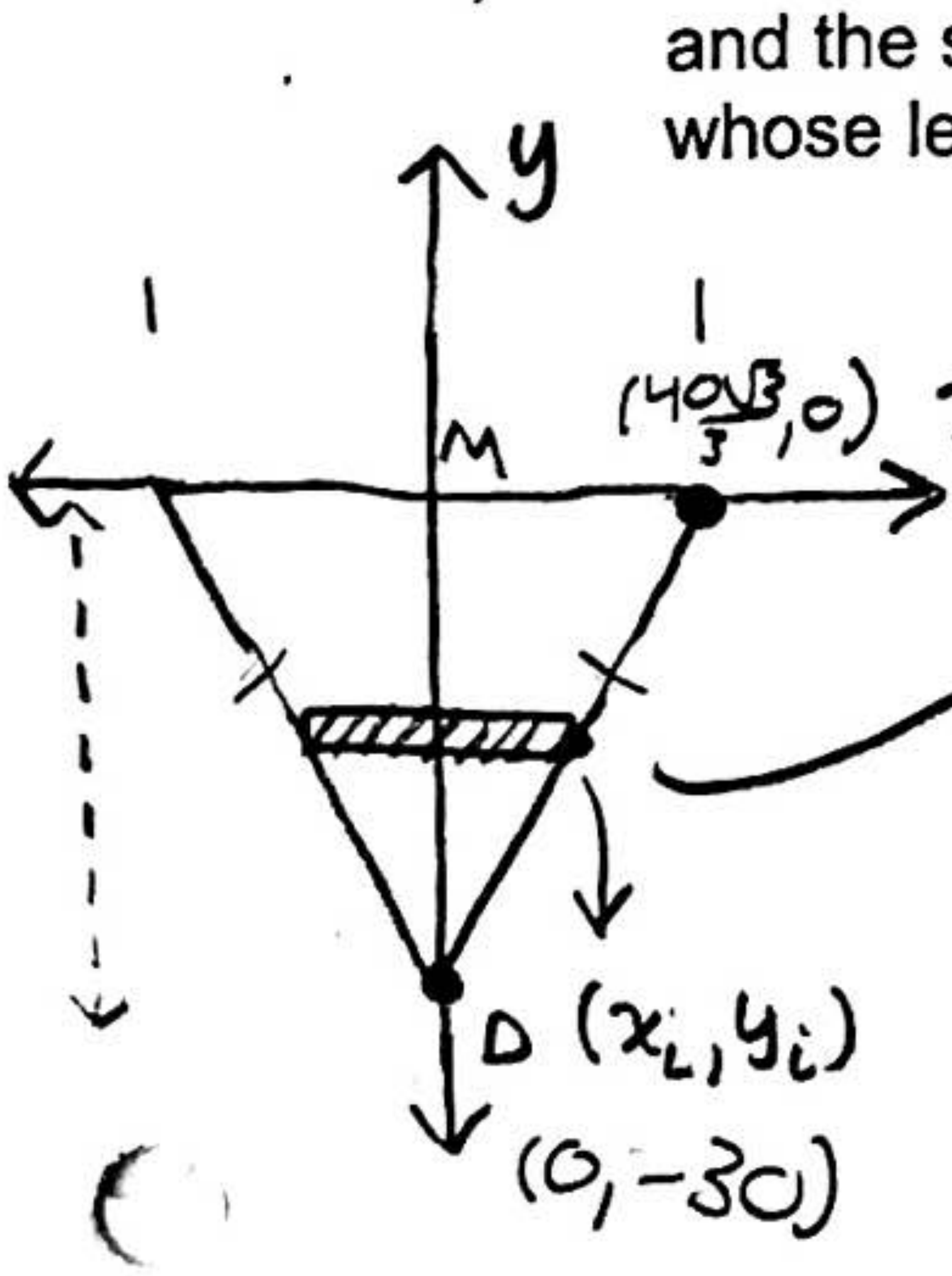




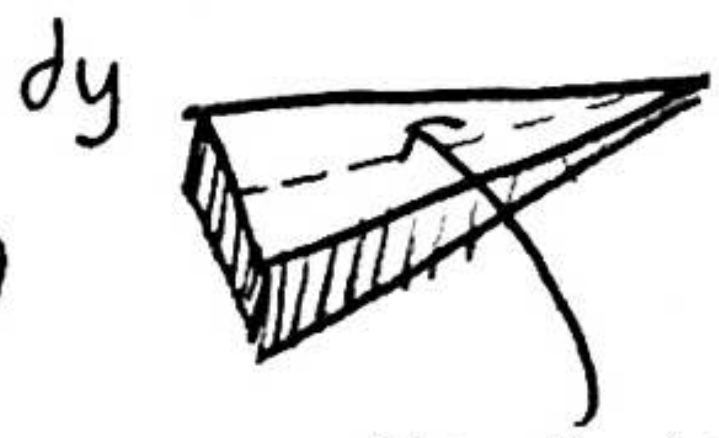


i) If this tank is full of water, then how much work is required to pump the water to the top, which is open, so that the water flows out?

ii) What if the top is closed and there is a spout at the top out from the water flows and the spout is 5 m tall? Assume that the tank is not full, but filled with water whose level is half the height of this tank.



$\overline{DM} = 30$ m



EQUALATERAL TRIANGLE SIMILAR TO ΔABC

IF b_i & h_i are the base HEIGHT OF THE SLICE THEN, $\dfrac{\overline{CM}}{\overline{AB}} = \dfrac{h_i}{b_i}$

$$\frac{\sqrt{3}}{80} \cdot \frac{40}{1} = \frac{h_i}{b_i}$$

$$\frac{\sqrt{3}}{2} = \frac{h_i}{b_i}$$

$$w = \int_0^{30} \frac{39{,}200\sqrt{3}}{9}\left(30y^2 - y^3\right)dy$$

$$= \frac{39{,}200\sqrt{3}}{9}\left(10y^3 - \frac{1}{4}y^4\right)\Big|_0^{30}$$

$$= \frac{39{,}200\sqrt{3}}{9}\left(270{,}000 - \frac{1}{4}\cdot 810{,}000\right)$$

$$= \frac{39{,}200\sqrt{3}}{9}(270{,}000 - 202{,}500)$$

$$= \frac{39{,}200\sqrt{3}}{9}\cdot 67{,}500$$

$$= 39{,}200\sqrt{3}\cdot 7{,}500$$

$$\boxed{= 294{,}000{,}000\sqrt{3}\text{ N-m or Joules}} \approx 509222937.4$$

$$\approx 5.09 \times 10^6$$

ii) What if the top is closed and there is a spout at the top out from the water flows and the spout is 5 m tall? Assume that the tank is not full, but filled with water whose level is half the height of this tank.

Let's reconsider what to change in the expression *:

$$w_i = \frac{39{,}200\sqrt{3}}{9}y_i^2\Delta y\cdot(35 - y_i)$$ (because the distance to the top of the spout has changed)

and now the integral to compute the total work done becomes:

$$w = \int_0^{15} \frac{39{,}200\sqrt{3}}{9}\left(35y^2 - y^3\right)dy$$

$$= \frac{39{,}200\sqrt{3}}{9}\left(\frac{35}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^{15}$$

$$= \frac{39{,}200\sqrt{3}}{9}\left(13{,}125 - \frac{1}{4}\cdot 1{,}875\right)$$

$$= \frac{39{,}200\sqrt{3}}{9}\cdot\frac{54{,}375 - 1{,}875}{4}$$

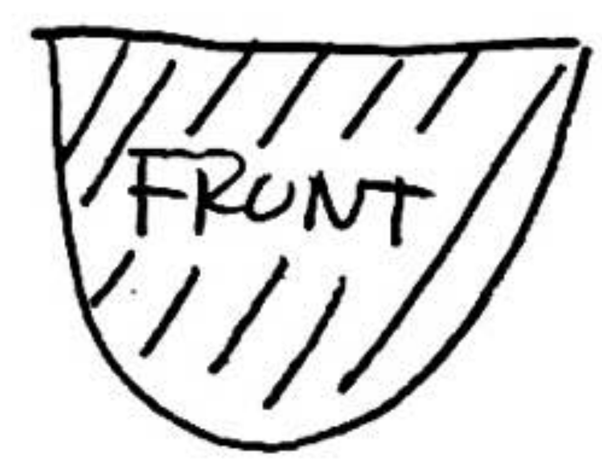
$$\boxed{= \frac{171{,}500{,}000\sqrt{3}}{3}\text{ N-m, or Joules}}$$

$$\approx 99015571.17$$

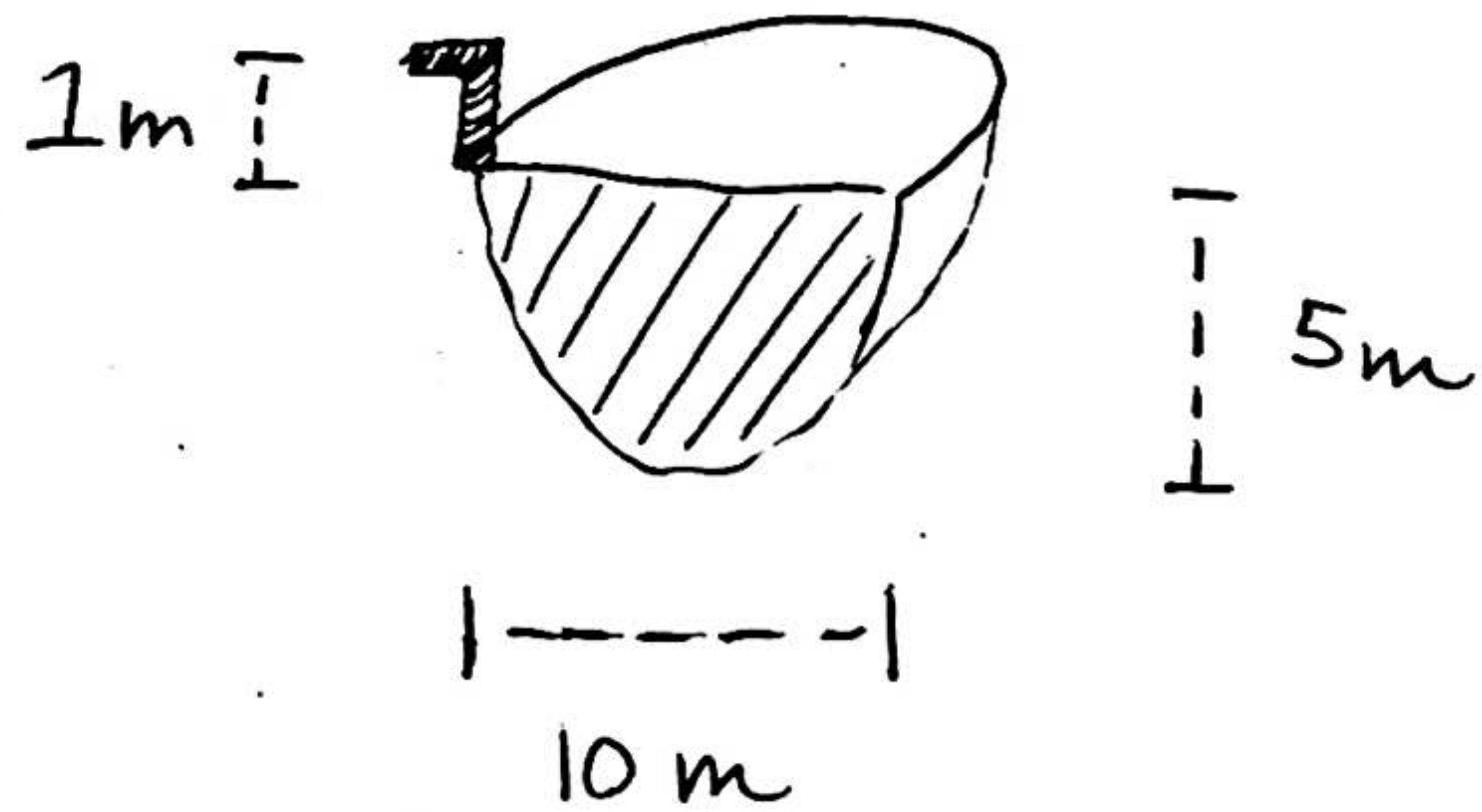
$$\approx 9.9 \times 10^7$$

§6.4

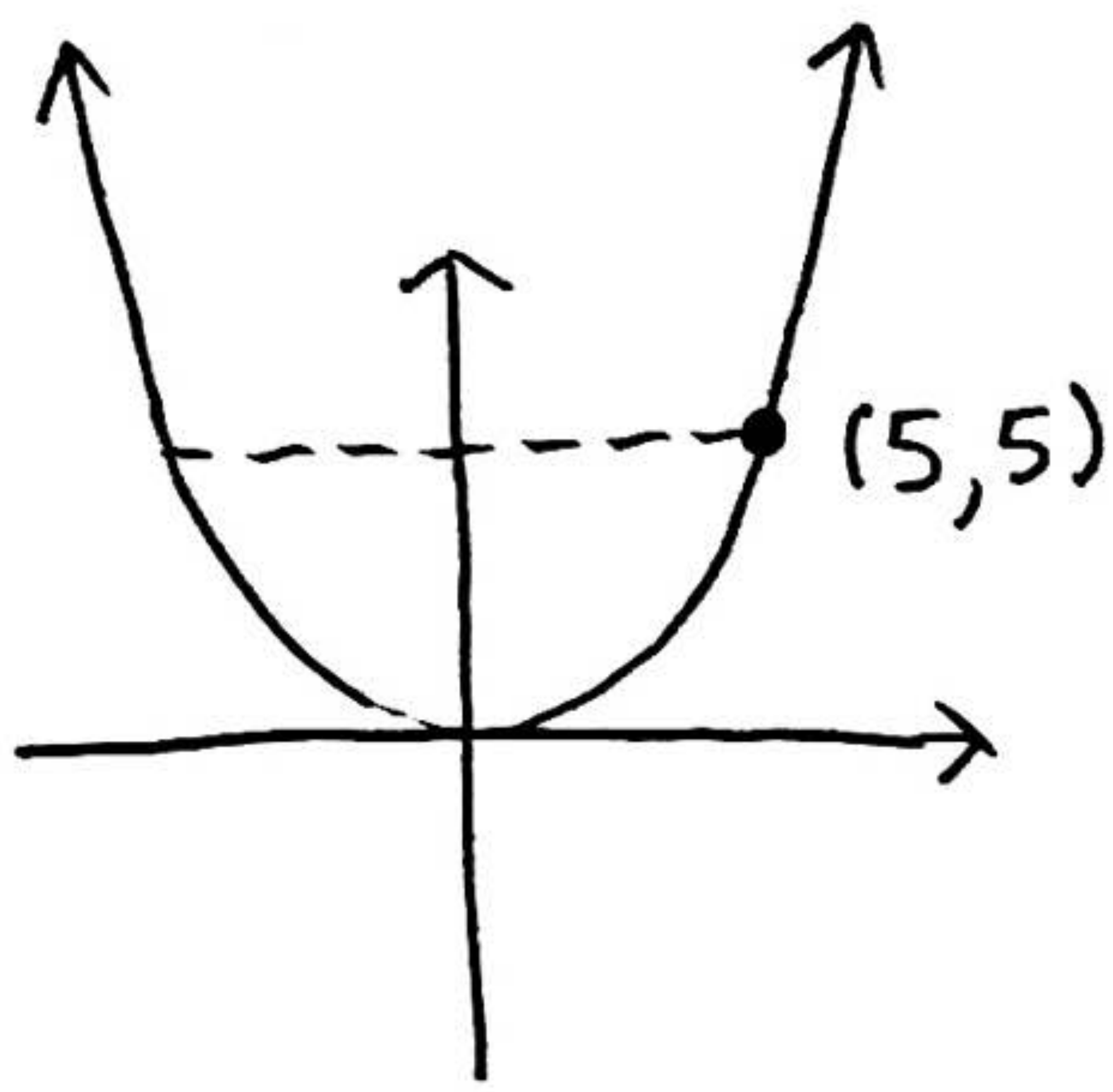
A TANK W/ A PARABOLIC FRONT 5 M HIGH & 10 M WIDE @ ITS TOP IS FILLED W/ WATER. 1 M ABOVE THE TANK SITS A SPOUT. IF TANK IS SEMI-CIRCULAR IF VIEWED FROM ABOVE, HOW MUCH WORK IS REQUIRED TO PUMP HALF OF THE WATER OUT OF THE TANK?



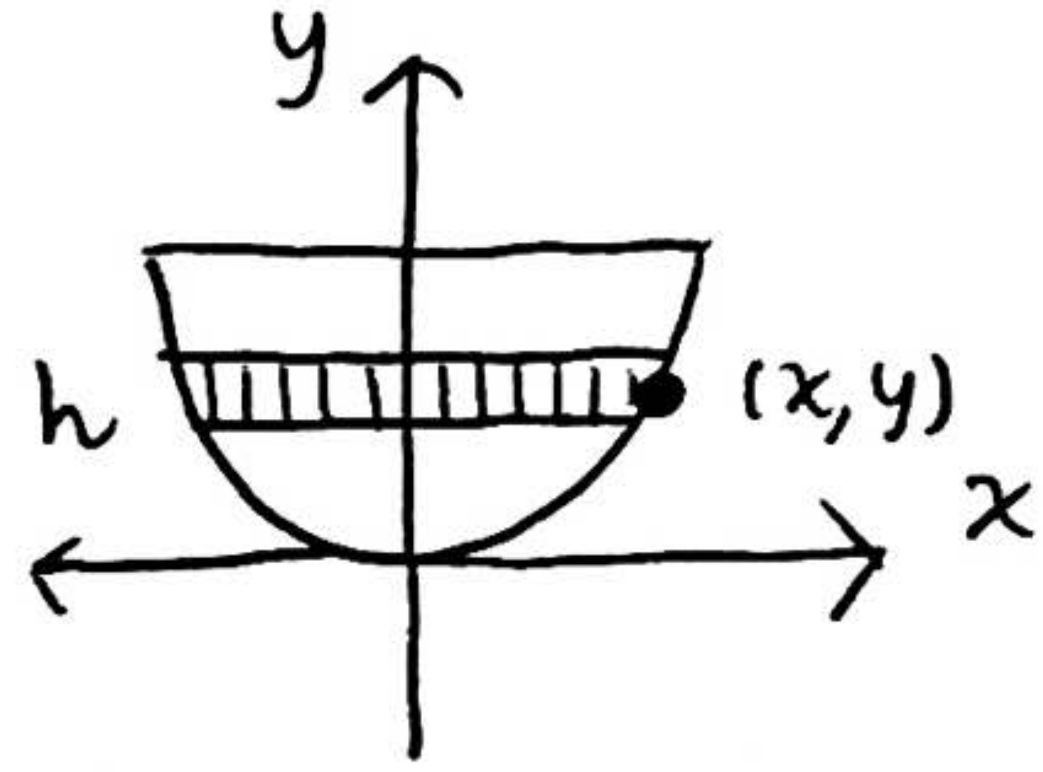
PARABOLA

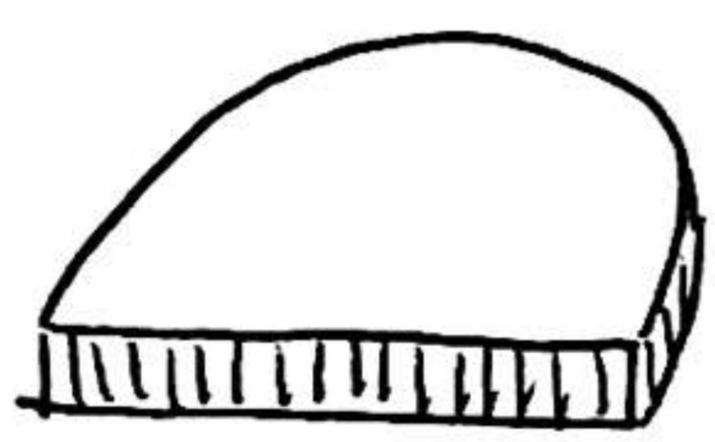


CIRCLE



$$y = ax^2$$
$$5 = a(5)^2$$
$$\therefore a = \frac{1}{5}$$
$$\therefore y = \frac{1}{5}x^2$$
or $x^2 = 5y$





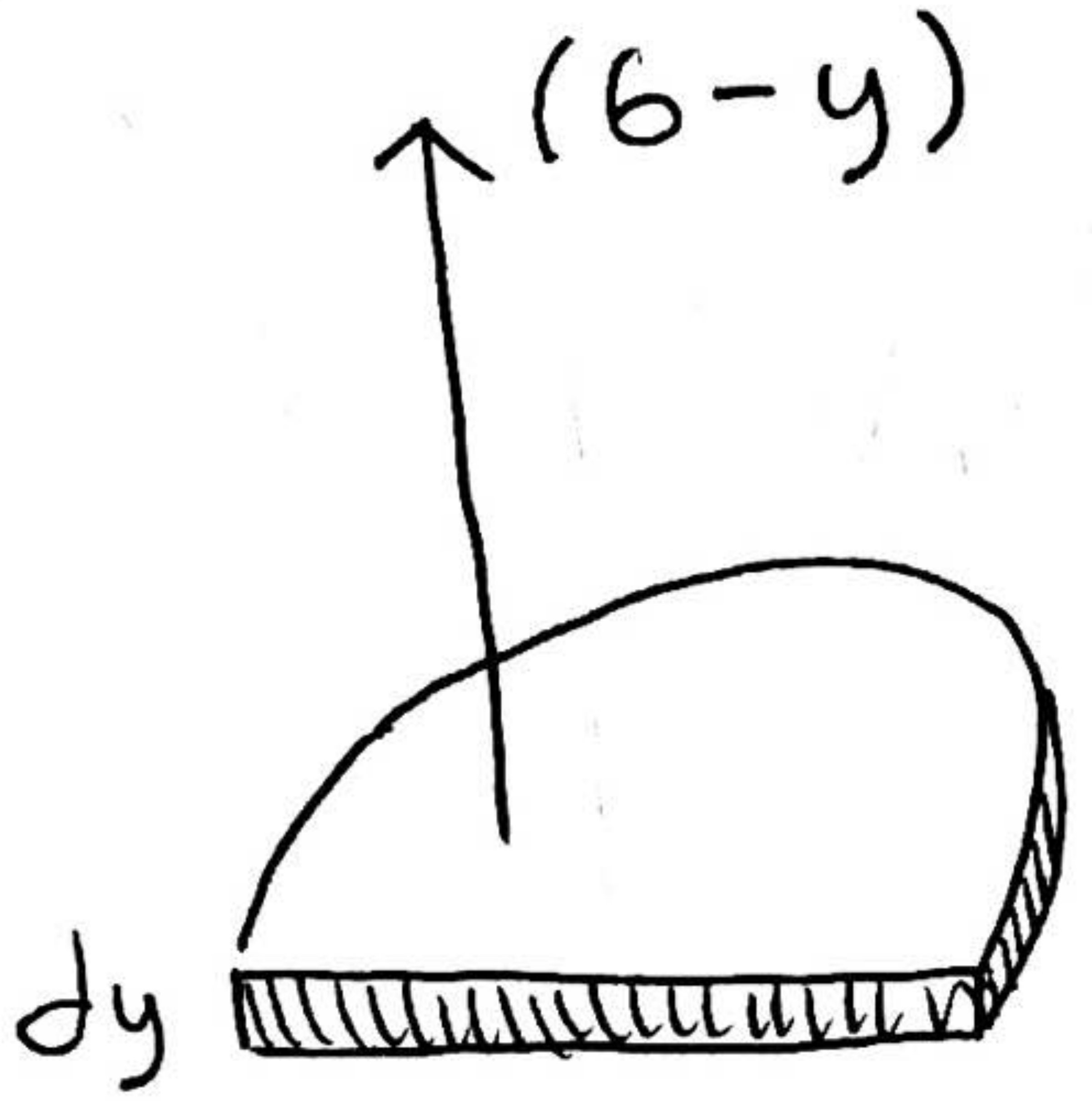
$$dV = A\,dy$$
$$= \tfrac{1}{2}\pi x^2\,dy$$
$$= \tfrac{1}{2}\pi\,5y\,dy$$

$$V = \int_0^5 \frac{\pi 5}{2} y\,dy$$
$$= \frac{5\pi}{2}\left[\frac{1}{2}y^2\right]_0^5$$
$$= \frac{5\pi}{2}\left[\frac{1}{2}25\right]$$
$$= \frac{125\pi}{4} \text{ UNITS}^3$$

TOTAL VOLUME

HALF-VOLUME IS $\frac{125\pi}{8}$ UNITS^3.

$$\frac{125\pi}{8} = \frac{5\pi}{2}\int_0^H y\,dy$$
$$\frac{125\pi}{8} = \frac{5\pi}{4}H^2$$
$$H^2 = \frac{25\pi}{2}$$
$$H = \frac{5\sqrt{\pi}}{\sqrt{2}} = \frac{5\sqrt{2\pi}}{2}$$



$$W = F \cdot D$$

$$dW = ma(6-y)$$

$$= \rho V g (6-y)$$

$$= \rho g \frac{5\pi}{2} y (6-y)\, dy$$

LET $\alpha = \frac{5\sqrt{2\pi}}{2}$

$$W = \int_0^\alpha \frac{5\pi}{2} \rho g y (6-y)\, dy$$

$$= \rho g \frac{5\pi}{2} \int_0^\alpha 6y - y^2\, dy$$

$$= \rho g \frac{5\pi}{2} \left[3y^2 - \frac{1}{3}y^3\right]_0^\alpha$$

$$= \rho g \frac{5\pi}{2} \left[3\alpha^2 - \frac{1}{3}\alpha^3\right]$$

$$= \rho g \frac{5\pi}{2} \frac{\alpha^2}{3} [9 - \alpha]$$

$$= \rho g \frac{5\pi}{6} \frac{25\pi}{2} \left[9 - \frac{5\sqrt{2\pi}}{2}\right]$$

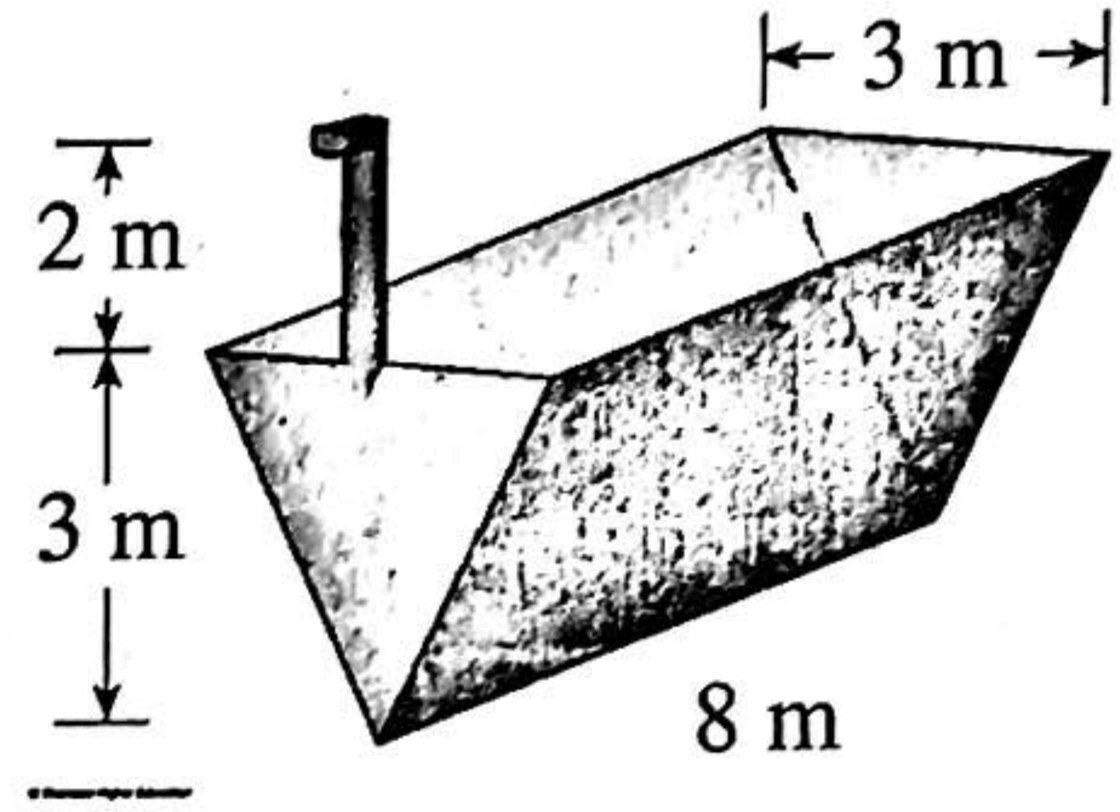
$$= 9800 \left(\frac{125\pi^2}{12}\right)\left(\frac{18 - 5\sqrt{2\pi}}{2}\right)$$

$$= \frac{1225}{3}(125\pi^2)(18 - 5\sqrt{2\pi})$$

$$\approx 2753990.5 \text{ J}$$

9.6.4

The tank below is full of water. Find the work required to pump the water out of the spout. Use the fact that water has a density of $1000 \, kg/m^3$.



$$= \int_0^3 (5-y)\, 9800\,(4y)\,dy$$

$$= 4(9800)\int_0^3 (20y - 4y^2)\,dy$$

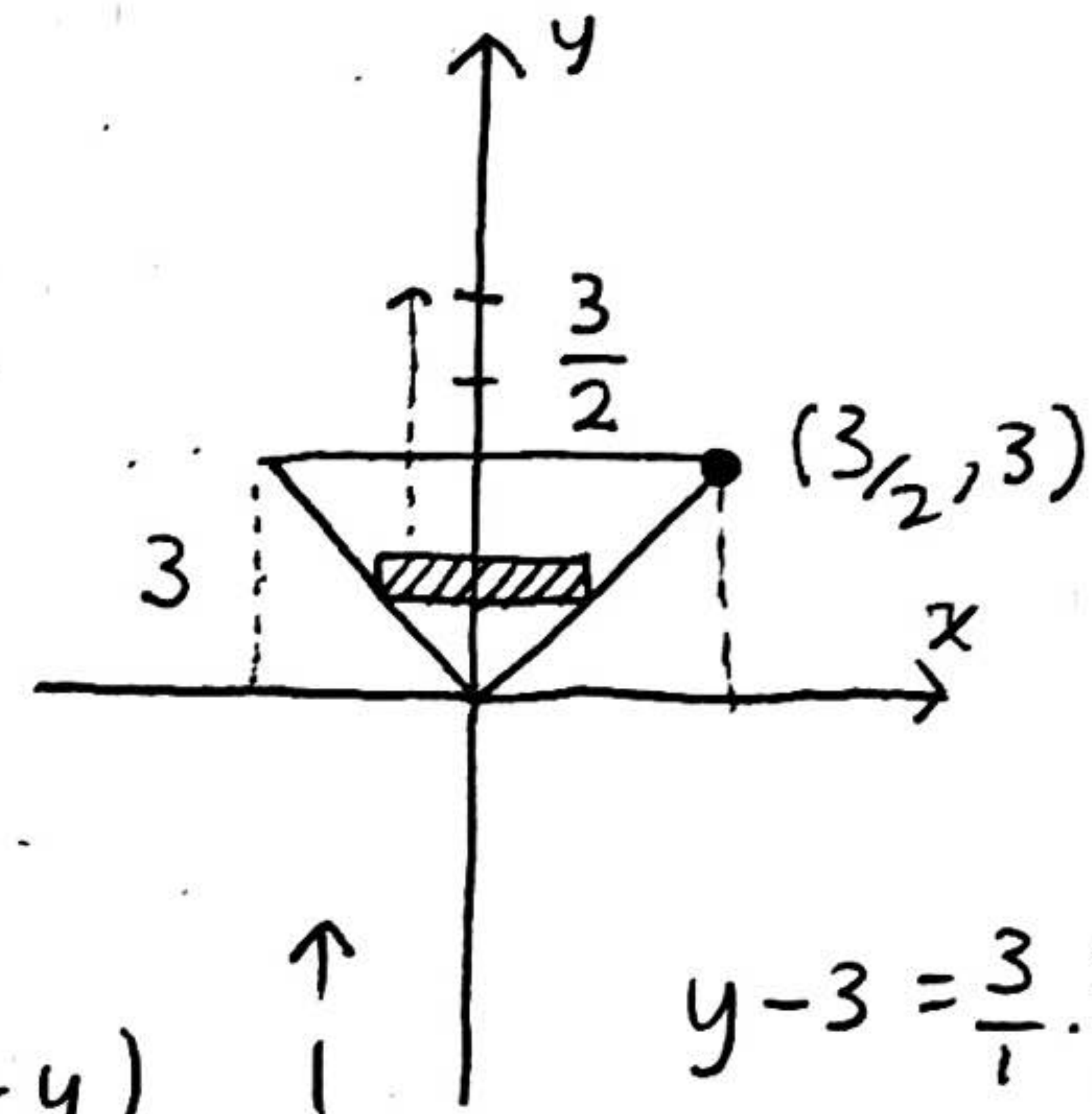
$$= 4(9800)\left[10y^2 - \frac{4}{3}y^3\right]_0^3$$

$$= 4(9800)\left[90 - \frac{4}{3}(27)\right]$$

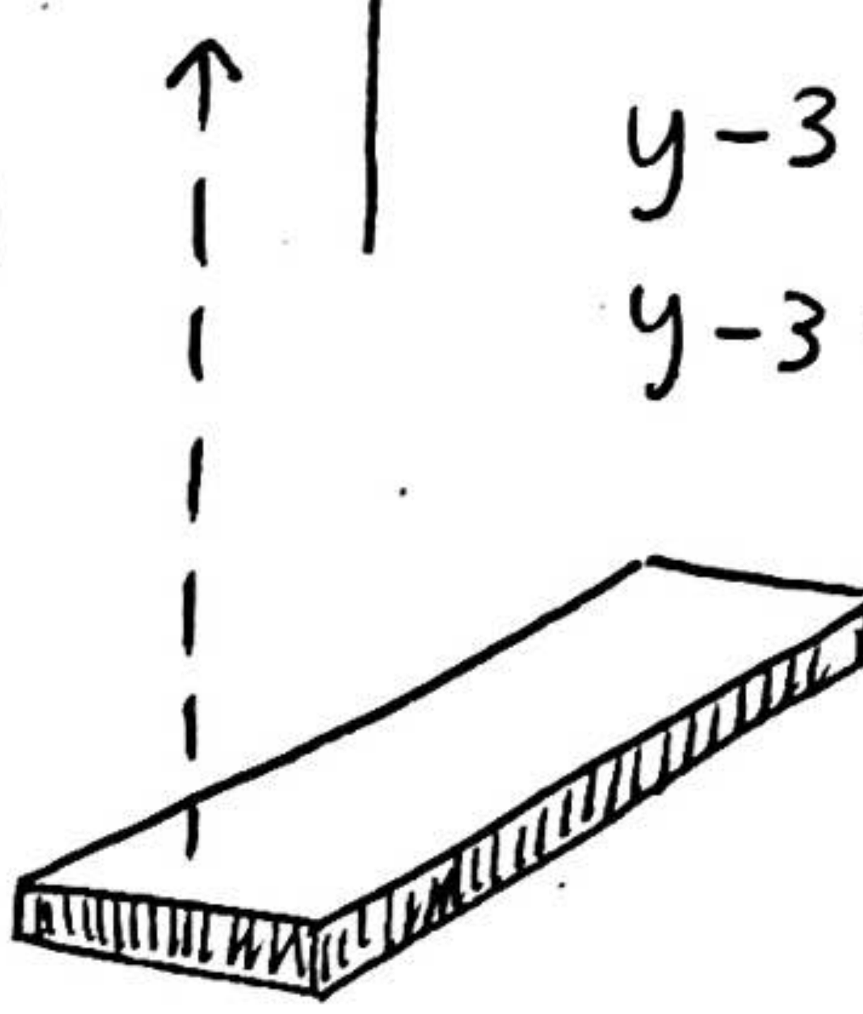
$$= 4(9800)\left[90 - 4(9)\right]$$

$$= 4(9800)(6.9)$$

$$= 4(9800)(54)$$



$(5-y)$



$$y-3 = \frac{3}{1}\cdot\frac{2}{3}\left(x-\frac{3}{2}\right)$$

$$y-3 = 2\left(x-\frac{3}{2}\right)$$

$$y = 2x$$

$$x = \frac{1}{2}y$$

$$V = hlw \quad (\times 2)$$

$$= 8f(y)\,dy$$

$$F = \rho V g$$

$$= 1000(8)f(y)\,dy \cdot g$$

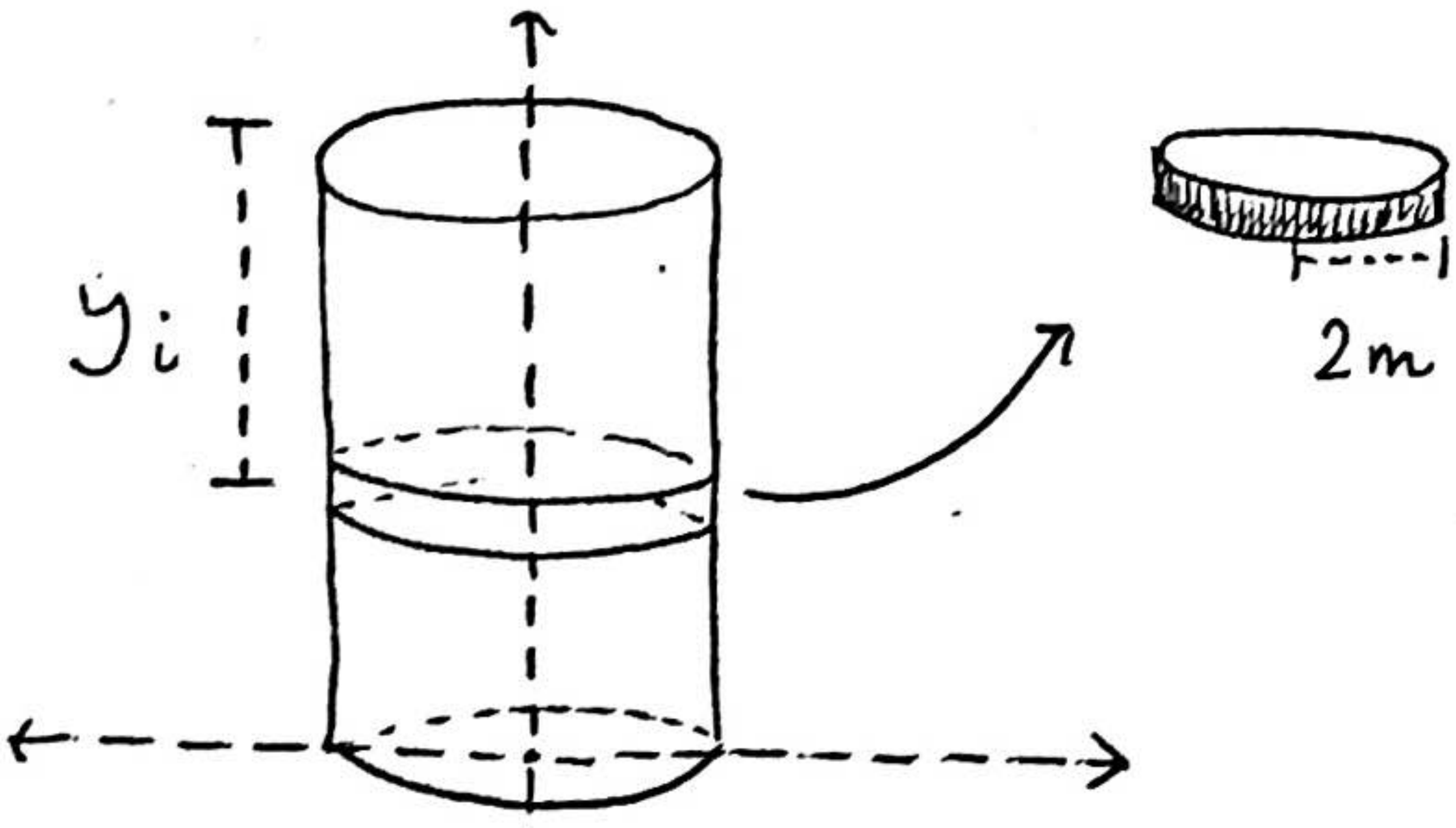
$$= 9800(8)\left(\frac{1}{2}y\right)dy\,(2)$$

$$= 9800(8)\,y\,dy$$

A cylindrical water tank has a height of 8 m and a radius of 2 m. Use the fact that water has a density of $1000\,\frac{kg}{m^3}$ 964

a. If the tank is full of water, determine how much $\boxed{\text{work}}$ is required to pump the water to the level of the top of the tank to empty the tank.

b. Is it true that it takes half as much work to pump the water out of the tank when it is only half full of water as when it is full? Explain.



$W_i = F \cdot d_i$

$= \rho V_i \cdot d_i$

$= \frac{1000\,kg}{m^3} \pi r^2 \, d \, \Delta y$

$= (1000)\pi(2)^2 \, d \, \Delta y$

$= 4000\pi \, (8 - y_i) \, \Delta y$

$W = \int_0^8 (4000\pi)(8-y)\,dy$

$W = 4000\pi \int_0^8 (8-y)\,dy$

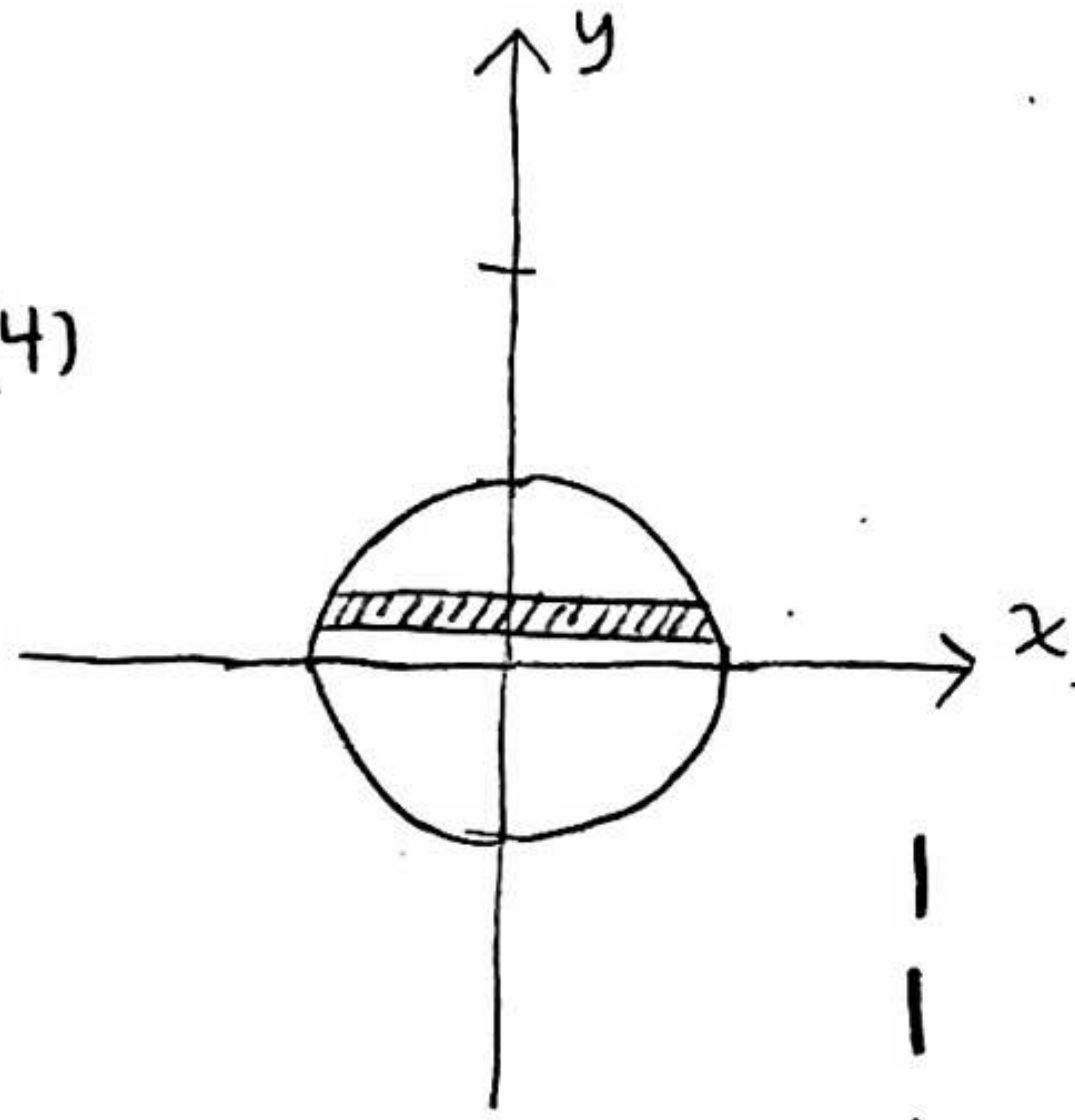
Name: ______________________________

Quiz 6: Work §6.4

1. A cylindrical gasoline drum is laid on its side. The drum has a radius of 1 m and length of 3 m. (assume gravity is 9.8 m/s^2 and that the density of water is 1000 kg / m^3)

   a. If the drum is full of water, determine how much work is required to pump all of water out of the tank from a spout is 1 m above the tank. (*Make sure to draw a picture that clearly defines your coordinate system and that identifies one of "slabs" of water you are moving*) (2) (2)

(4)



$$W_1 = \int_{-1}^{1} 9.8(1000)(2-y)(6\sqrt{1-y^2})\,dy$$

EVEN ODD

$$= 6(9800)\int_{-1}^{1} 2\sqrt{1-y^2} - y\sqrt{1-y^2}\,dy$$

$$= 24(9800)\int_0^1 \sqrt{1-y^2}\,dy$$

$$= 2.352(10^5)\int_0^1 \sqrt{1-y^2}\,dy$$

$$= 2.352(10^5)\int_{x=0}^{x=1} \cos\theta\cos\theta\,d\theta$$

$$= 2.352(10^5)\int_{x=0}^{x=1} \frac{\cos 2\theta + 1}{2}\,d\theta$$

$$= K\left[-\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\frac{\pi}{2}}$$

$$= K\left[-\frac{1}{4}\sin\left(2\left(\frac{\pi}{2}\right)\right) + \frac{1}{2}\left(\frac{\pi}{2}\right)\right]$$

$$= K\left[-\frac{\pi}{4}\right] = \boxed{58{,}800\pi}$$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$2\cos^2\theta - 1 = \cos 2\theta$

$y = \sin\theta$

$dy = \cos\theta$

$\theta = \frac{\pi}{2}$ $\theta =$

$x^2 + y^2 = 1$

$x^2 = 1 - y^2$

$x = \pm\sqrt{1-y^2}$

$y_1 = \sqrt{1-x^2}$

$y_2 = -\sqrt{1+x^2}$

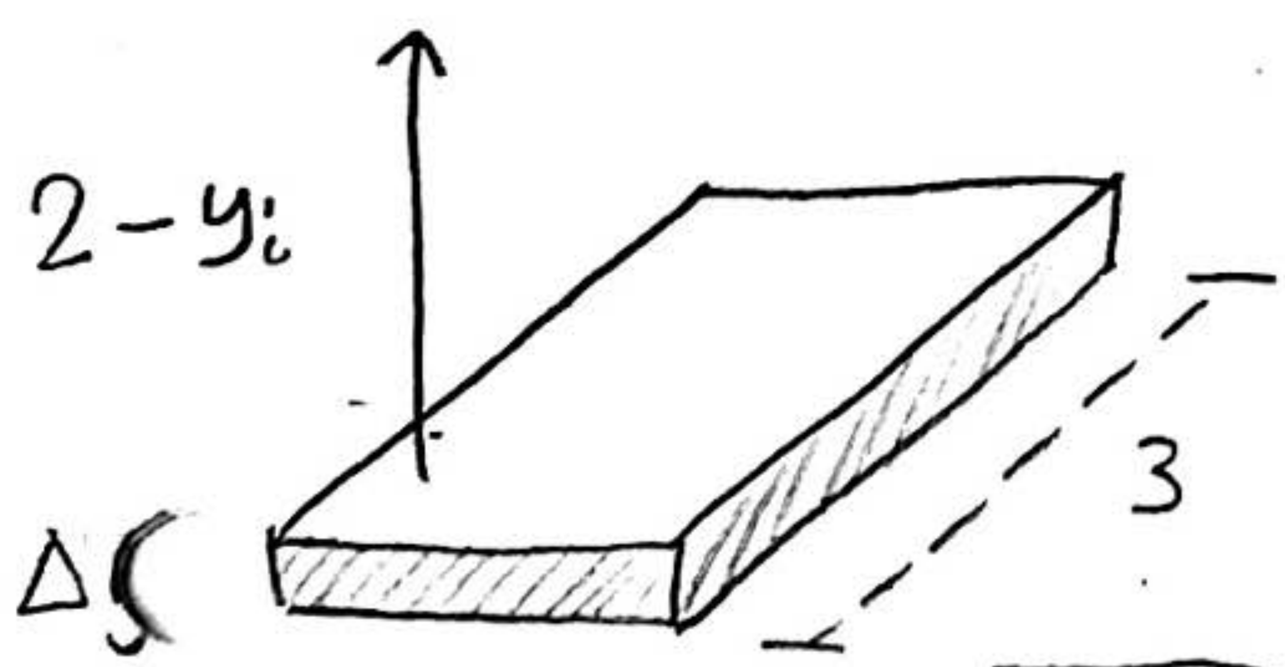
$W = FD$

$= maD$

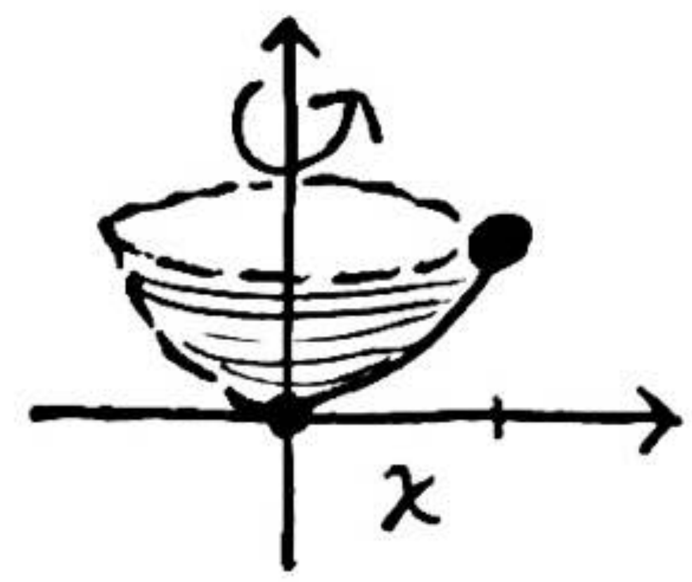
$= \rho V a D$

$= (1000)(\Delta y_i)(3)(2x_i)(2-y_i)(9.8)$

$= 9800(6)\Delta y_i(2-y_i)\sqrt{1-y_i}$



$2x_i = \pm 2\sqrt{1-y_i^2}$



EXAMPLE 1

$$y = \frac{1}{3}x^{3/2}$$

$$3y = x^{3/2}$$

$$9y^2 = x^3$$

$$18y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{x^2}{6(\frac{1}{3}x^{3/2})} = \frac{x^2}{2x^{3/2}} = \frac{1}{2}x^{\frac{1}{2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}x$$

$$L = 2\pi\int_0^1 x\sqrt{1+\tfrac{1}{4}x}\,dx$$

$$= 2\pi\int_1^{5/4}(4u-4)\sqrt{u}\,(4)\,dx$$

$$= 32\pi\int_1^{5/4} u^{3/2} - u^{\frac{1}{2}}\,du$$

$$= 32\pi\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_1^{5/4}$$

$$= \frac{\pi}{15}\left[\frac{128 - 10\sqrt{5}}{1}\right]$$

$$u = 1 + \frac{1}{4}x$$

$$du = \frac{1}{4}dx$$

$$4\,du = dx$$

$$u(0) = 1$$

$$u(1) = \frac{5}{4}$$

$$4u - 4 = x$$

§8.1

98.3

or 98.2

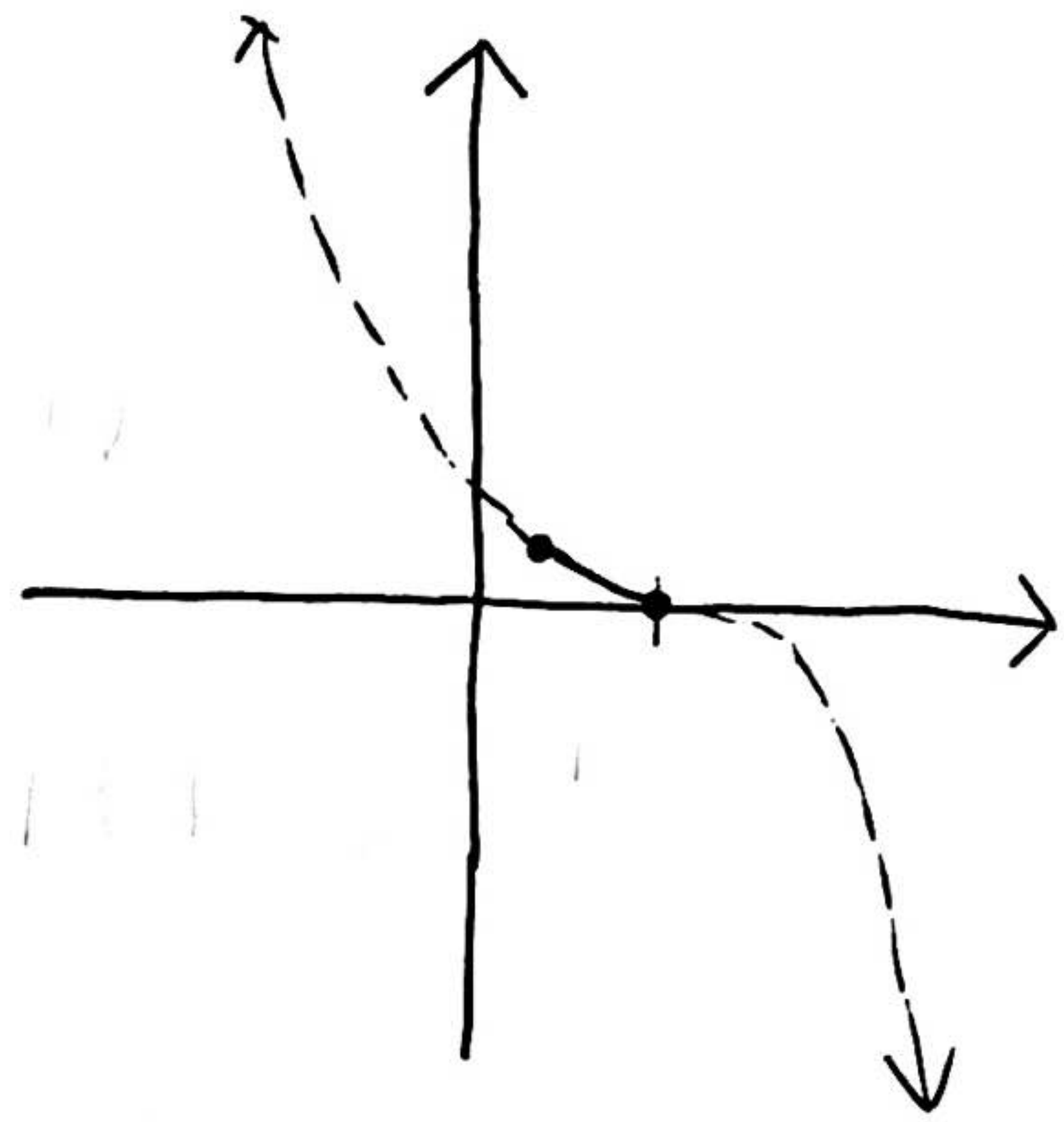
$$y = (1-x)^3$$

$$\sqrt[3]{y} = 1 - x$$

$$x = 1 - \sqrt[3]{y}$$

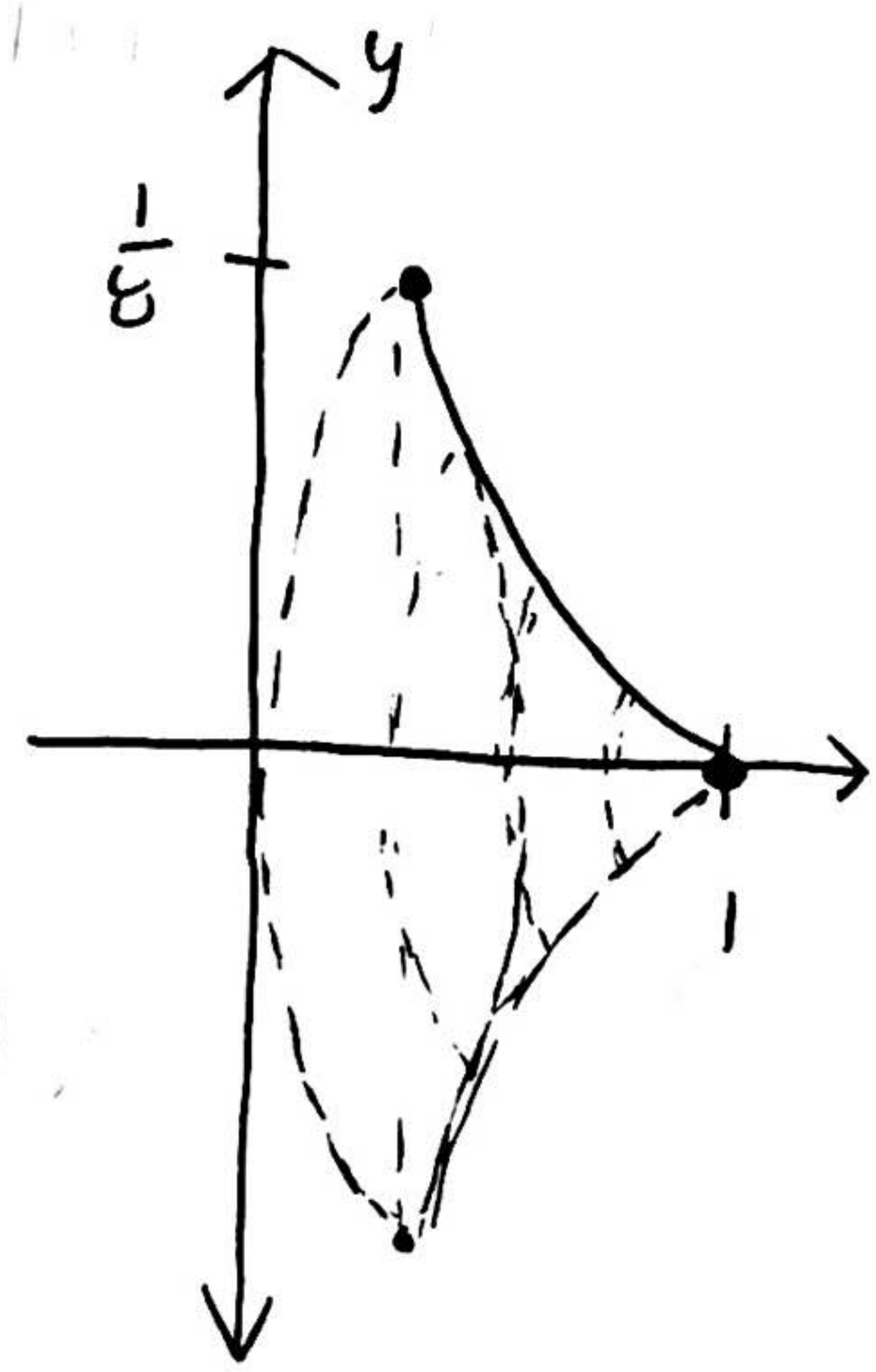
$$= 1 - y^{\frac{1}{3}}$$

$$x' = -\frac{1}{3} y^{-\frac{2}{3}}$$



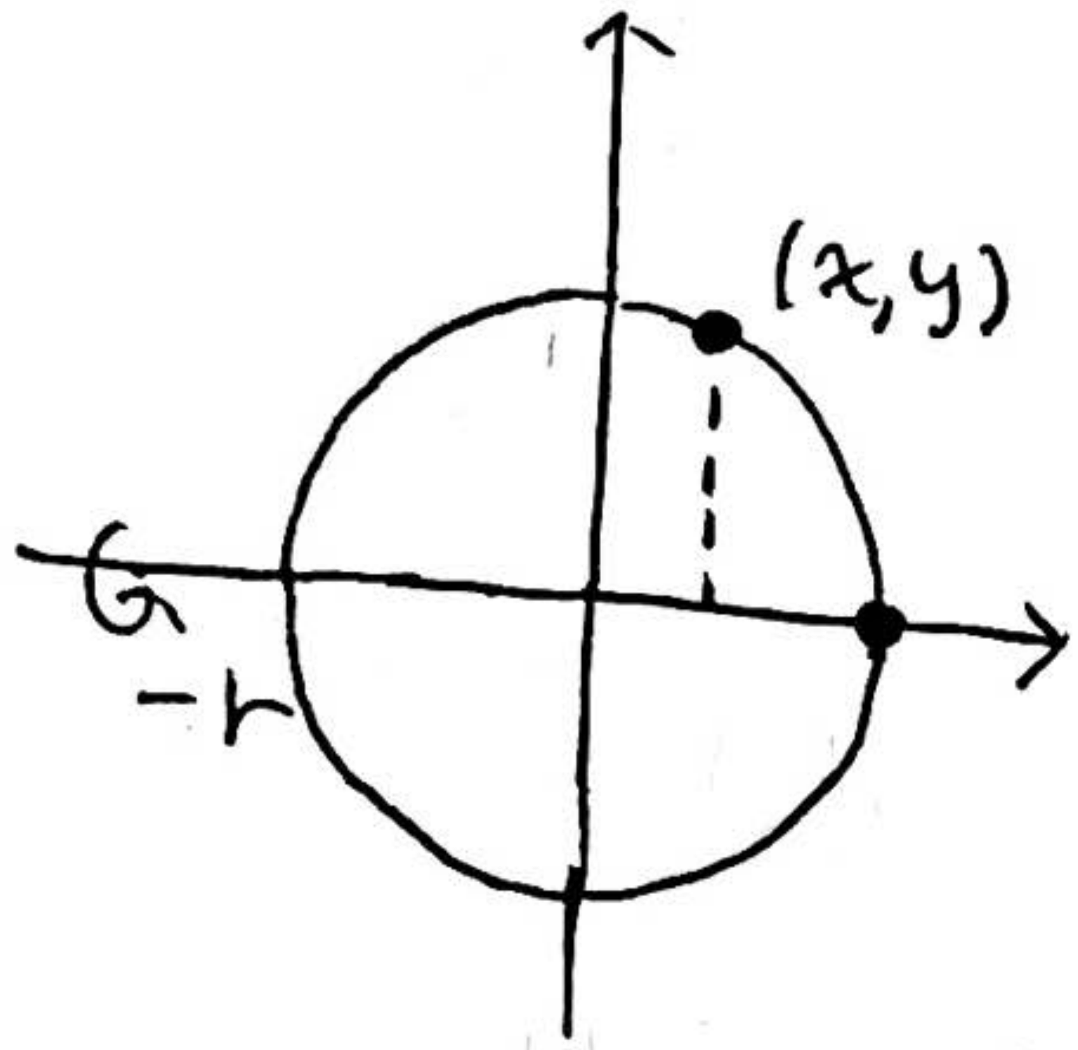
$$1 + (x')^2 = 1 + \frac{1}{9} y^{-\frac{4}{3}}$$

$$y = -(x-1)^3$$



ITP

SURFACE AREA OF A CIRCLE



$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$$

$$S = 2\pi \int_{-r}^{r} y \, ds$$

$$= 2\pi \int_{-r}^{r} y \sqrt{1 + \frac{x^2}{r^2-x^2}} \, dx$$

$$= 4\pi \int_0^r \sqrt{r^2+x^2} \sqrt{\frac{r^2}{r^2-x^2}} \, dx$$

$$= 4\pi \int_0^r r \, dx$$

$$= 4\pi x r \Big|_0^r = 4\pi r^2$$

$\boxed{\S 8.2}$

E3

§ 8.2

$$S = 2\pi \int_0^1 x \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\, dx$$

$$= 2\pi \int_0^1 x \sqrt{(2x)^2 + 1}\, dx$$

$$= 2\pi \int_0^1 x \sqrt{4x^2 + 1}$$

$$A = \frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

$y = x^2$ $\qquad \dfrac{dy}{dx} = 2x$

$x = \sqrt{y}$ $\qquad \dfrac{dx}{dy} = \dfrac{1}{2} y^{-\frac{1}{2}}$

$x(0) = \sqrt{0} = 0$

$x(1) = \sqrt{1} = 1$

$\boxed{E2}$

V.2

$$S = 2\pi \int_0^1 x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\pi \int_0^1 x \sqrt{1 + \frac{1}{4y}}\, dy$$

$$= 2\pi \int_0^1 \sqrt{y} \sqrt{1 + \frac{1}{4y}}\, dy$$

$$= 2\pi \int_0^1 \sqrt{y + \frac{1}{4}}\, dy$$

$$= \pi \int_0^1 \sqrt{4y+1}\, dy$$

$\boxed{E3}$?

§8.2

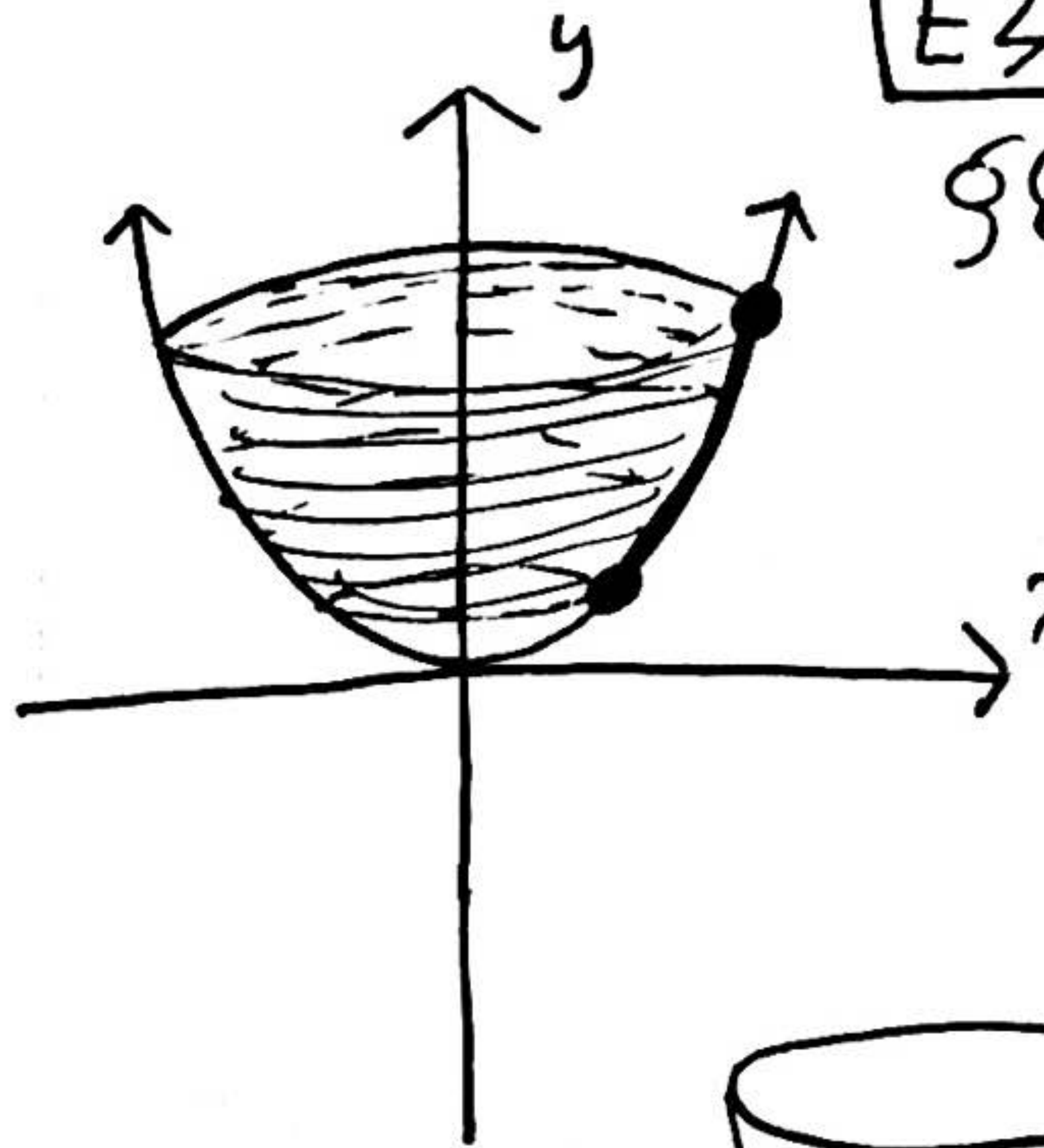
$y = x^2$

$x = \sqrt{y}$

$1 + f'(y)^2$

$= 1 + (\sqrt{y})^2$

$= 1 + y$



$$S = 2\pi \int_0^1 \sqrt{y}\sqrt{1+y}\, dy$$

$$= 2\pi \int_0^1 \sqrt{y^2+y}\, dy$$

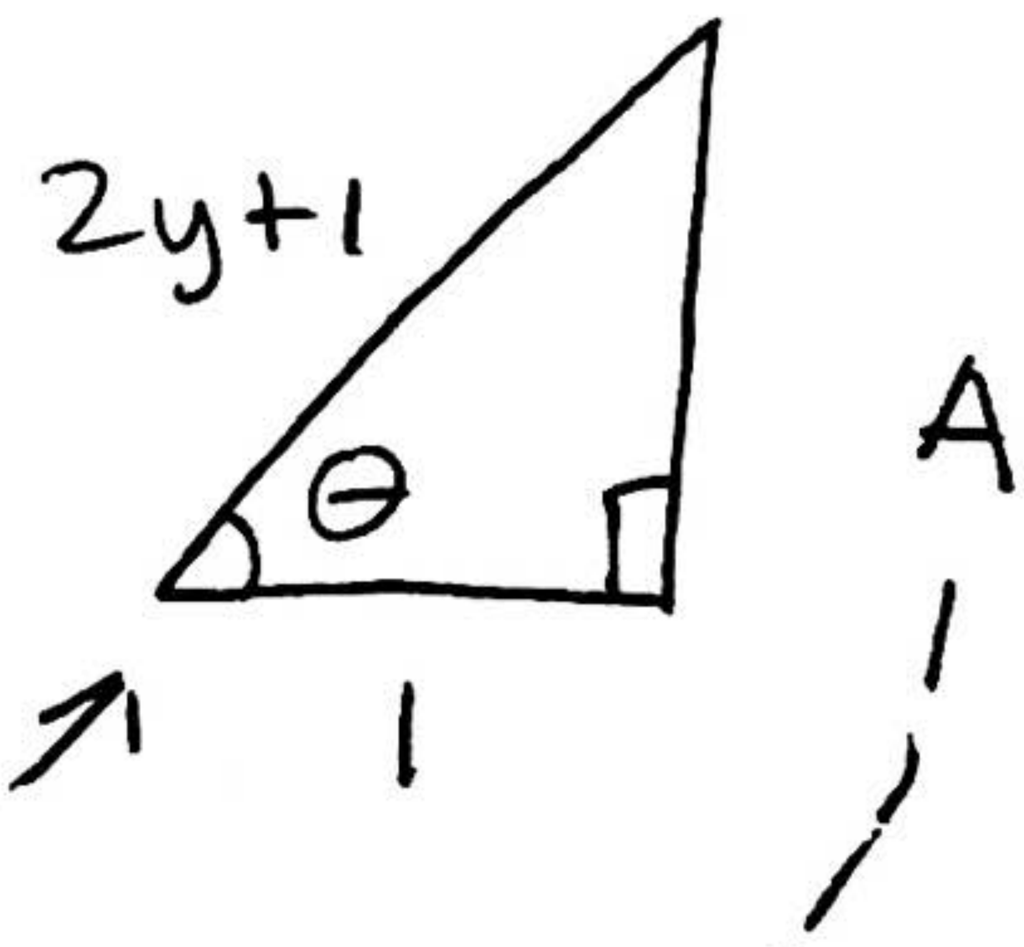
$$= 2\pi \int_0^1 \sqrt{y^2+y+\tfrac{1}{4}-\tfrac{1}{4}}\, dy$$

$$= 2\pi \int_0^1 \sqrt{(y+\tfrac{1}{2})^2-\tfrac{1}{4}}\, dy$$

$y + \frac{1}{2} = \frac{1}{2}\sec\theta$

$dy = \frac{1}{2}\sec\theta\tan\theta\, d\theta$

$2y+1 = \sec\theta$



$A = \sqrt{4y^2+4y+1-1}$

$= 2\sqrt{y^2+y}$

$\boxed{E2}$

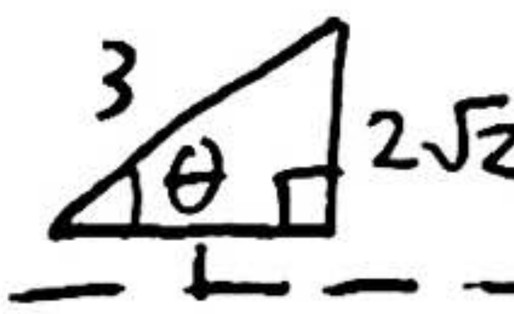
§8.2

$$S = 2\pi \int \sqrt{\tfrac{1}{4}\sec^2\theta - \tfrac{1}{4}}\left(\tfrac{1}{2}\sec\theta\tan\theta\right) d\theta$$

$$= 2\pi \int \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\sqrt{\tan^2\theta}\,\sec\theta\tan\theta\, d\theta$$

$1 + \frac{1}{2} = \frac{1}{2}\sec\theta$

$\sec\theta = 3$



$y + \frac{1}{2} = \frac{1}{2}\sec\theta$

$0 + \frac{1}{2} = \frac{1}{2}\sec\theta$

$\sec\theta = 1$

$\theta = 0$

$$= \frac{\pi}{2}\int \tan^2\theta\,\sec\theta\, d\theta$$

$$= \frac{\pi}{2}\int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \frac{\pi}{2}\int \sec^3\theta - \sec\theta\, d\theta$$

$$= \frac{\pi}{2}\left[\frac{\sec\theta\tan\theta}{2} - \frac{\ln(\tan\theta + \sec\theta)}{2}\right]\Bigg|_0^{\sec^{-1}(3)}$$

$$= \frac{\pi}{2}\left[\frac{3(2\sqrt{2})}{1} - \frac{\ln(2\sqrt{2}+3)}{2} - 0 + 0\right]$$

$$= \frac{\pi}{2}\left[6\sqrt{2} - \frac{1}{2}\ln(2\sqrt{2}+3)\right]$$

EXAMPLE 2

WHAT IS THE LENGTH OF ONE SINE CURVE?

$$L = \int_0^{2\pi} \sqrt{1+\cos^2 x}\, dx$$

THIS INTEGRAL IS "IMPOSSIBLE" TO DO. BUT THIS LENGTH IS USED IN SCIENCE ALL THE TIME.

WHAT ABOUT THE SURFACE AREA?

WHY π?

$$S = 2\pi \int_0^{\pi} \sin(x)\sqrt{1+\cos^2 x}\, dx$$

$$u = \cos x \qquad du = -\sin x\, dx$$

$$S = -2\pi \int_1^{-1} \sqrt{1+u^2}\, du$$

$$= \cancel{2}\pi \int_0^1 \sqrt{1+u^2}\, du$$

WHAT TWO THINGS HAPPENED HERE?

$$= \cancel{2}\pi \int_0^{\pi/4} \sec^2\theta \sqrt{1+\tan^2\theta}\, d\theta$$

$\sec^3\theta$ AGAIN?!?

$$= \cancel{2}\pi \int_0^{\pi/4} \sec^3\theta\, d\theta$$

$$= \cancel{2}\pi \left[\frac{1}{4}\ln\left(\frac{1}{2}\right) + \frac{1}{\sqrt{2}}\right] = \pi\left[\ln\left(\frac{\sqrt{2}}{2}\right) + \sqrt{2}\right]$$

$$y = \sqrt{4-x^2} - 1$$

$$-1 \le x \le 1$$

$$f(x) = \sqrt{4-x^2} - 1$$

$$f'(x) = \frac{-2x}{2\sqrt{4-x^2}}$$

$$= \frac{-x}{\sqrt{4-x^2}}$$

$$1 + [f'(x)]^2$$

$$= 1 + \frac{x^2}{4-x^2}$$

$$= \frac{4-x^2+x^2}{4-x^2}$$

$$= \frac{4}{4-x^2}$$

$$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= 2\pi \int_{-1}^{1} (\sqrt{4-x^2}-1)\sqrt{\frac{4}{4-x^2}}\,dx$$

$$= 4\pi \int_{-1}^{1} 1 - \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= 8\pi \int_0^1 1\,dx - 8\pi \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= 8\pi\left[x - \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1\right]$$

$$= 8\pi\left[1 - \sin^{-1}\left(\frac{1}{2}\right) + 0\right]$$

$$= 8\pi\left[1 - \frac{\pi}{6}\right]$$

$$= \boxed{\frac{8\pi[6-\pi]}{6}}$$

$$\approx 11.973\ldots$$

$$\approx 12$$

SURFACE AREA
§8.2

$$S_2 = 2\pi \int_1^2 x \sqrt{\frac{1+9x^{4/3}}{x^{4/3}}}\, dx$$

$$= 2\pi \int_1^2 x \left(\frac{1}{x^{2/3}}\right) \sqrt{1+9x^{4/3}}\, dx$$

$$= 2\pi \int_1^2 x^{1/3} \sqrt{1+9x^{4/3}}\, dx$$

LOOK! An "easy" SUBSTITUTION!
$(u = 1 + 9x^{4/3})$

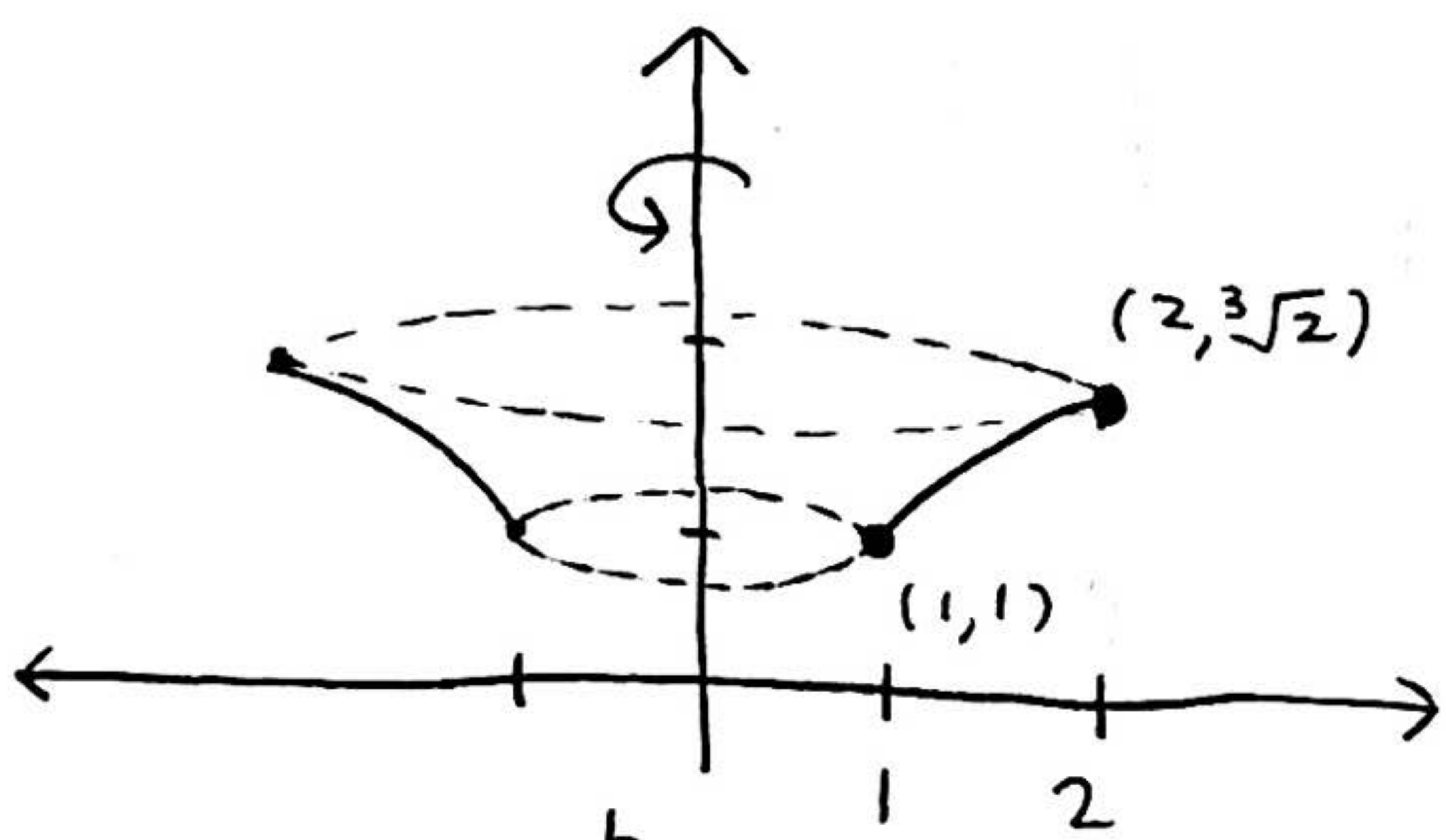
$$= 2\pi \int_{x=1}^{x=2} \left(\frac{1}{12}\right) \sqrt{u}\, du$$

$$= \frac{\pi}{6} \left[\frac{2}{3} u^{3/2} \right]_{x=1}^{x=2}$$

$$= \frac{\pi}{6} \cdot \frac{2}{3} \left(1+9x^{4/3}\right)^{3/2} \Big|_1^2$$

$$= \frac{\pi}{9} \left[\left(1+18\sqrt[3]{2}\right)^{3/2} - 10^{3/2} \right].$$

FIND THE SURFACE AREA OF $y = \sqrt[3]{x}$ FOR $1 \le x \le 2$ ROTATED ABOUT Y-AXIS



$$S = \int_a^b 2\pi x \, ds$$

$y = \sqrt[3]{x}$; $1 \le x \le 2$

$x = y^3$; $1 \le y \le \sqrt[3]{2}$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$ $\quad \frac{dx}{dy} = 3y^2$

$$S = \int_a^b 2\pi x \, ds$$

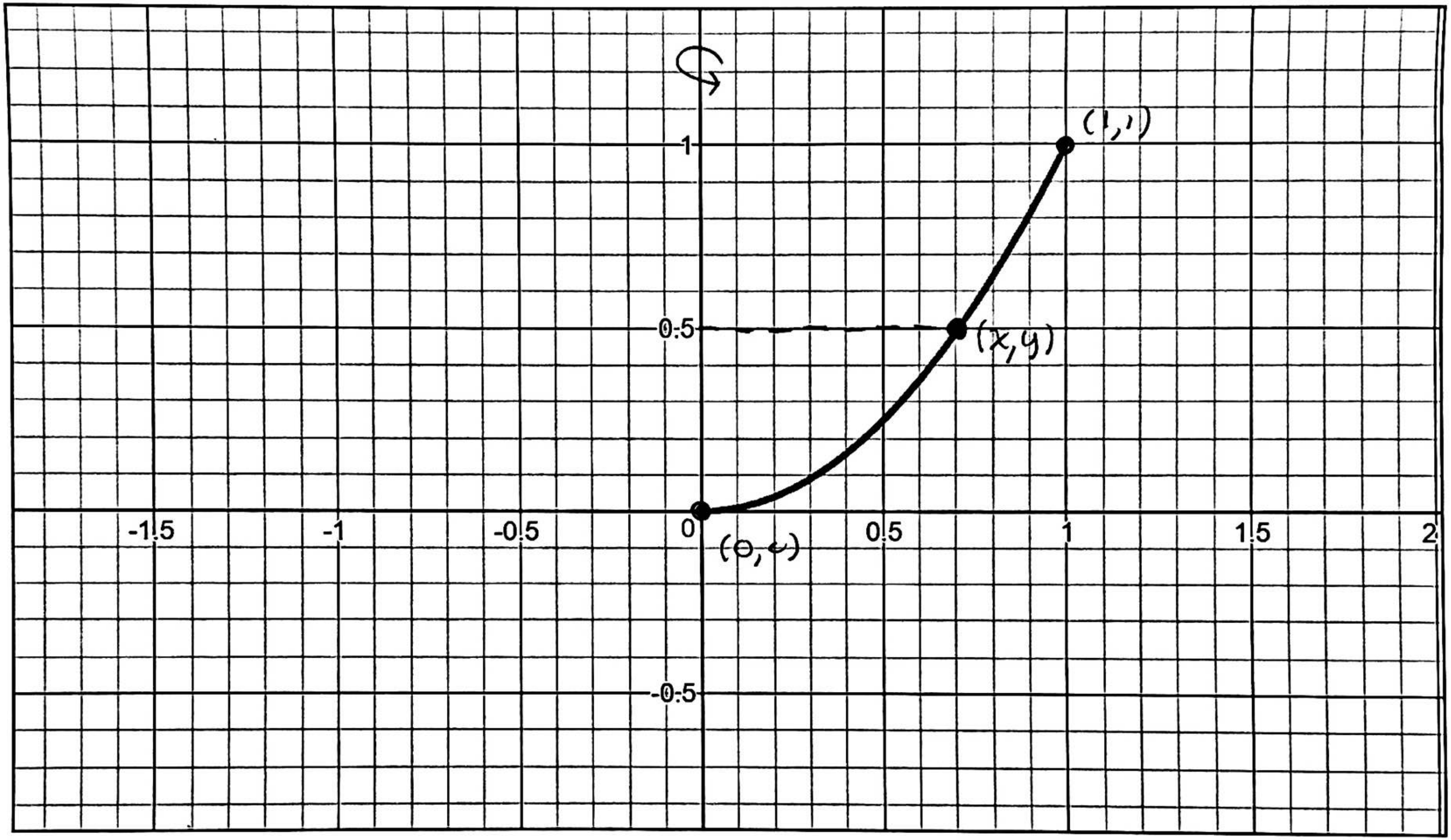
$$S_1 = 2\pi \int_1^{\sqrt[3]{2}} x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= 2\pi \int_1^{\sqrt[3]{2}} y^3 \sqrt{1 + 9y^4} \, dy$$ ---> EASY U-SUB

$$S_2 = \int_1^2 2\pi x \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx$$

$$= 2\pi \int_1^2 x \sqrt{\frac{1}{9}x^{-4/3} + 1} \, dx$$

EXAMPLE 3



$$S = 2\pi \int_{y=0}^{y=1} g(y)\sqrt{(g'(y))^2+1}\,dy = 2\pi \int_{x=0}^{x=1} f(x)\sqrt{1+f'(x)^2}\,dx$$

E4

§8.2

$$y = e^x$$

$$1 + (y')^2 = 1 + e^{2x}$$

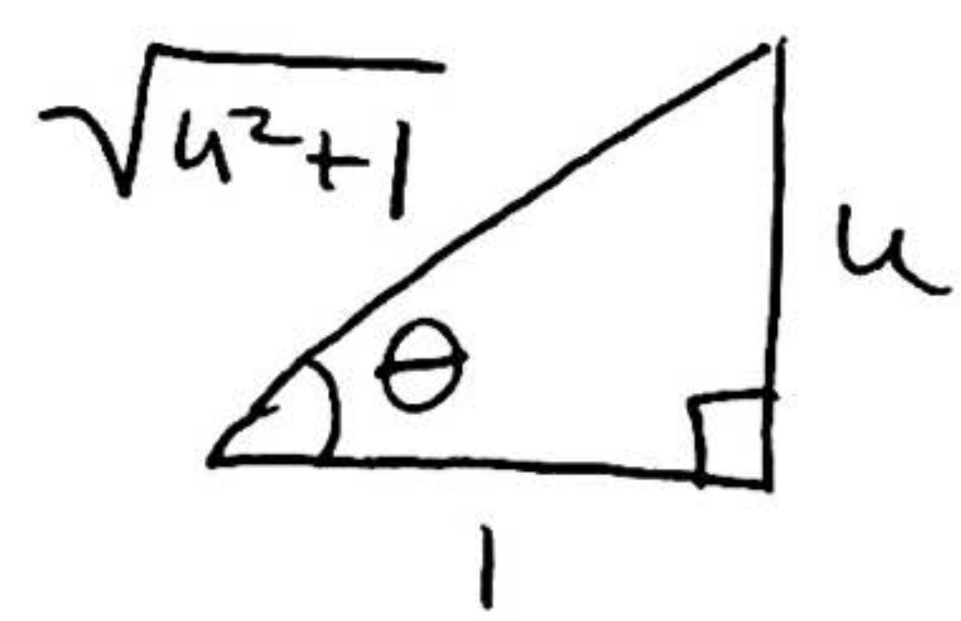
$$S = 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx$$

$$u = e^x \qquad u^2 = e^{2x}$$

$$du = e^x\, dx \qquad u(0) = 1 \qquad u(1) = e$$

$$S = 2\pi \int_1^e \sqrt{1 + u^2}\, du \qquad \begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{array}$$

$$= 2\pi \int_{u=1}^{u=e} \sec^3\theta\, d\theta$$



$$= 2\pi \left[\tfrac{1}{2}\right] \Big(\ln|\sec\theta + \tan\theta| + \sec\theta\tan\theta \Big) \Big|_{u=1}^{u=e}$$

$$= \pi \left[\ln\left|\sqrt{u^2+1} + u\right| + u\sqrt{u^2+1} \right] \Big|_1^e$$

$$= \pi \left[\ln(\sqrt{e^2+1} + e) + e\sqrt{e^2+1} - \ln(\sqrt{2}+1) + \sqrt{2} \right]$$

1

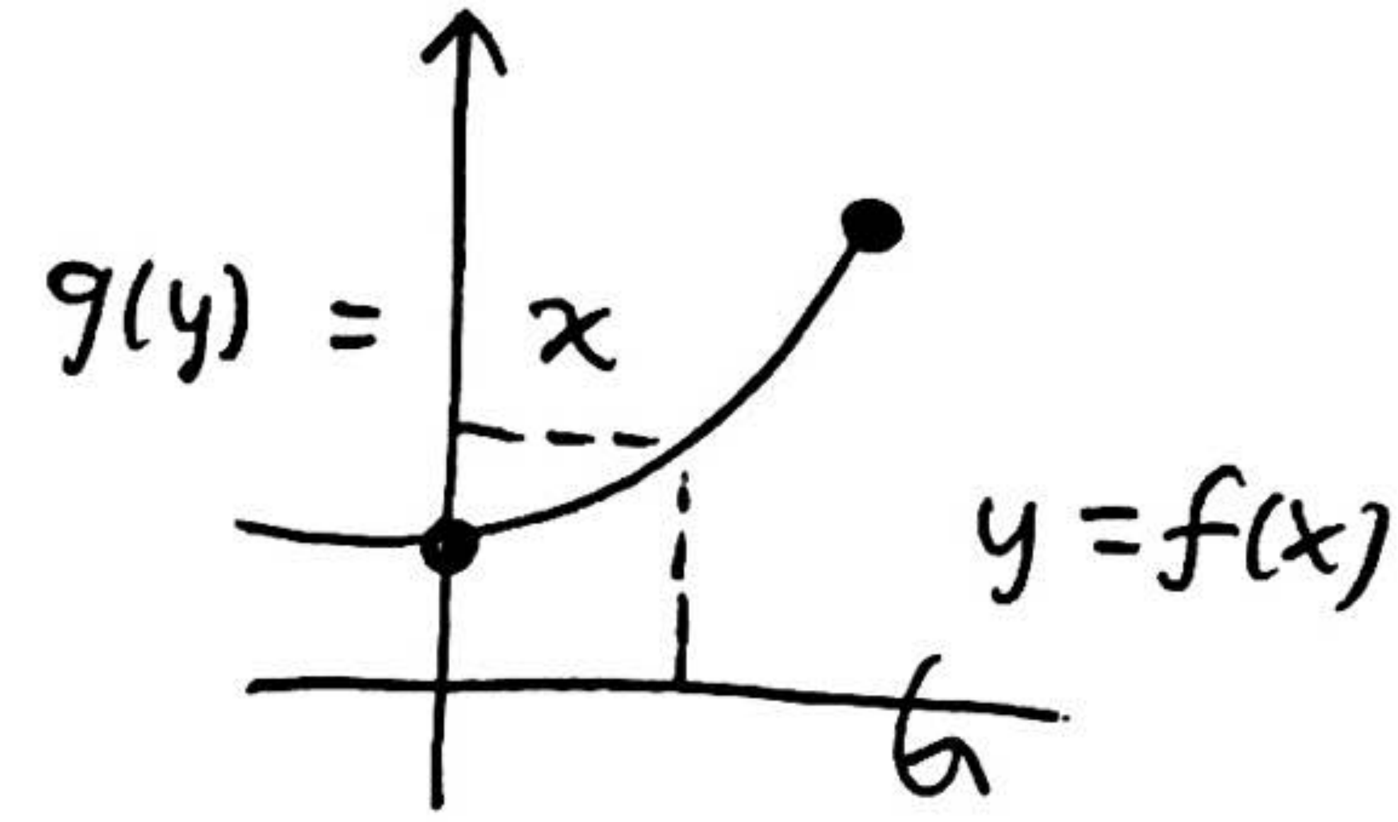
$$x = \ln y$$

$$\frac{\partial x}{\partial y} = \frac{1}{y}$$

$$2\pi \int_1^e y\sqrt{1+\frac{1}{y^2}}\, \partial y$$

$$= 2\pi \int_1^e \frac{y\sqrt{y^2+1}}{y}\, \partial y$$

$$= 2\pi \int_1^e \sqrt{y^2+1}\, \partial y$$



$y = e^x$

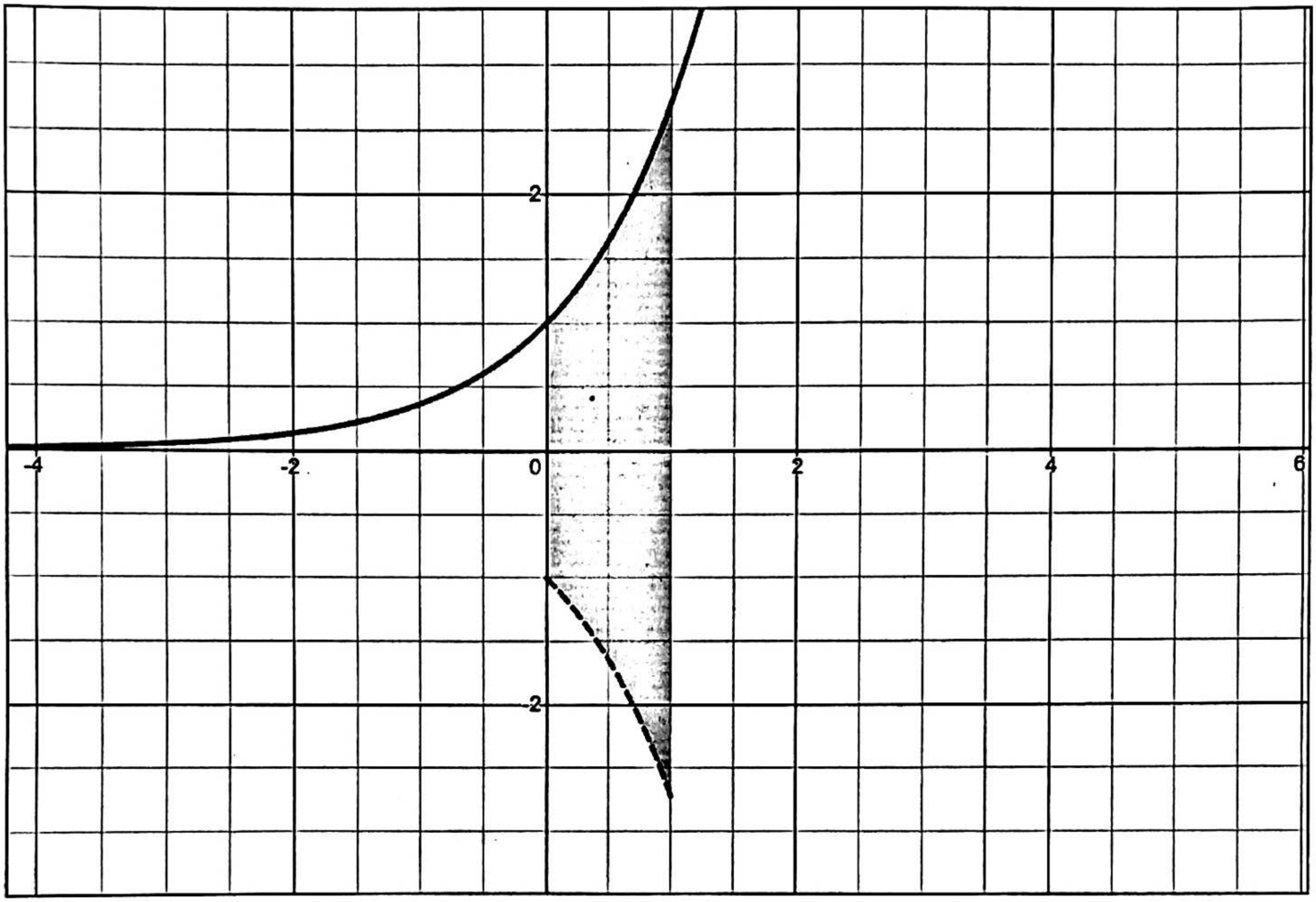
$\ln y = x$

$$\frac{dx}{dy} = \frac{1}{y}$$

$$S = 2\pi \int_1^e \ln y \sqrt{1 + \left(\tfrac{1}{y}\right)^2}\, dy$$

$$= 2\pi \int_1^e \ln y \sqrt{\frac{y^2+1}{y^2}}\, dy$$

$$= 2\pi \int_1^e \frac{1}{y} \ln y \sqrt{y^2+1}\, dy$$



EXAMPLE 4

$\boxed{\S 8.2}$

$$L = \int_a^b \sqrt{1+f'(x)^2}\, dx$$

$$L = \int_\alpha^\beta \sqrt{1+\tan^2\theta}\, \sec^2\theta \, d\theta$$

$$= \int_\alpha^\beta |\sec\theta| \sec^2\theta \, d\theta$$

$$= \int_\alpha^\beta \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\Big[\ln|\tan\theta + \sec\theta| + \sec\theta\tan\theta\Big]\Big|_\alpha^\beta$$

$$= \frac{1}{2}\ln\left|b + \sqrt{b^2+1}\right| + \frac{1}{2}b\sqrt{b^2+1} - \frac{1}{2}\ln\left|a + \sqrt{a^2+1}\right| - \frac{1}{2}a\sqrt{a^2+1}$$

$$= \frac{1}{2}\left[\ln\left|\frac{b+\sqrt{b^2+1}}{a+\sqrt{a^2+1}}\right| + b\sqrt{b^2+1} - a\sqrt{a^2+1}\right]$$

WHY DOESN'T THIS WORK?

MAYBE A BETTER QUESTION IS, WHAT DOES THIS IMPLY ABOUT ANY ARCLENGTH?

$\mathrm{Dom}(\tan^{-1}) = \mathbb{R}$
$\mathrm{Rng}(\tan^{-1}) = (-\frac{\pi}{2}, \frac{\pi}{2})$

$$u = f'(x)$$

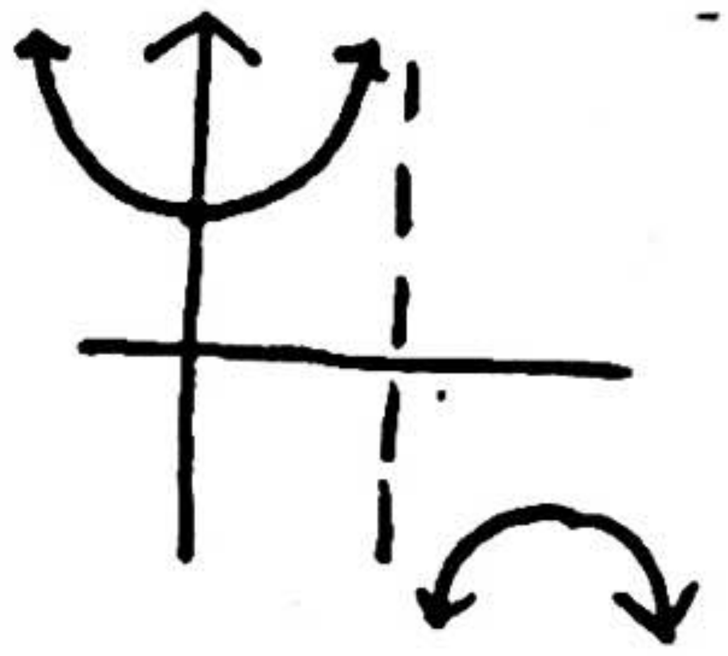
$$u = \tan\theta$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{du}{d\theta} = \sec^2\theta$$

$$du = \sec^2\theta\, d\theta$$

$$\sec\theta > 0$$

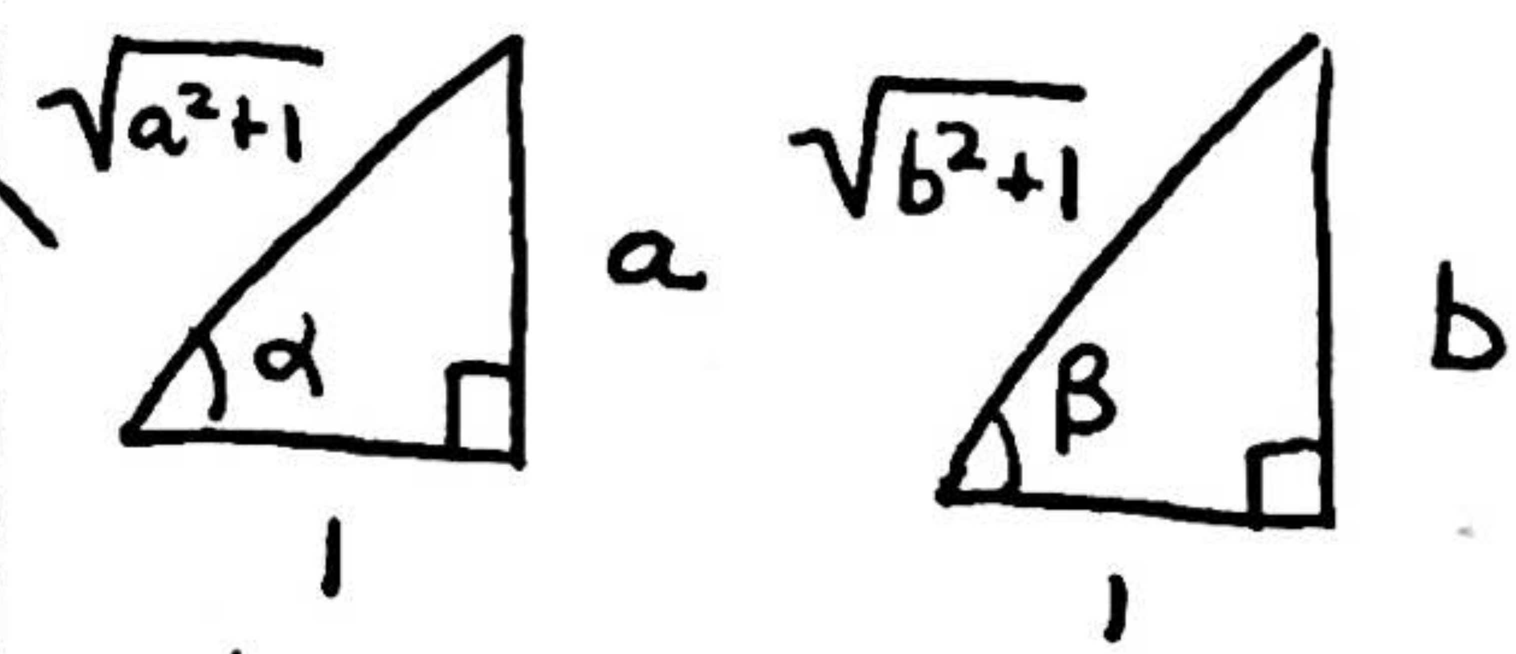


$\sec\big|_{(-\frac{\pi}{2}, \frac{\pi}{2})}$
RANGE $[1, \infty)$

$$a = \tan\alpha$$

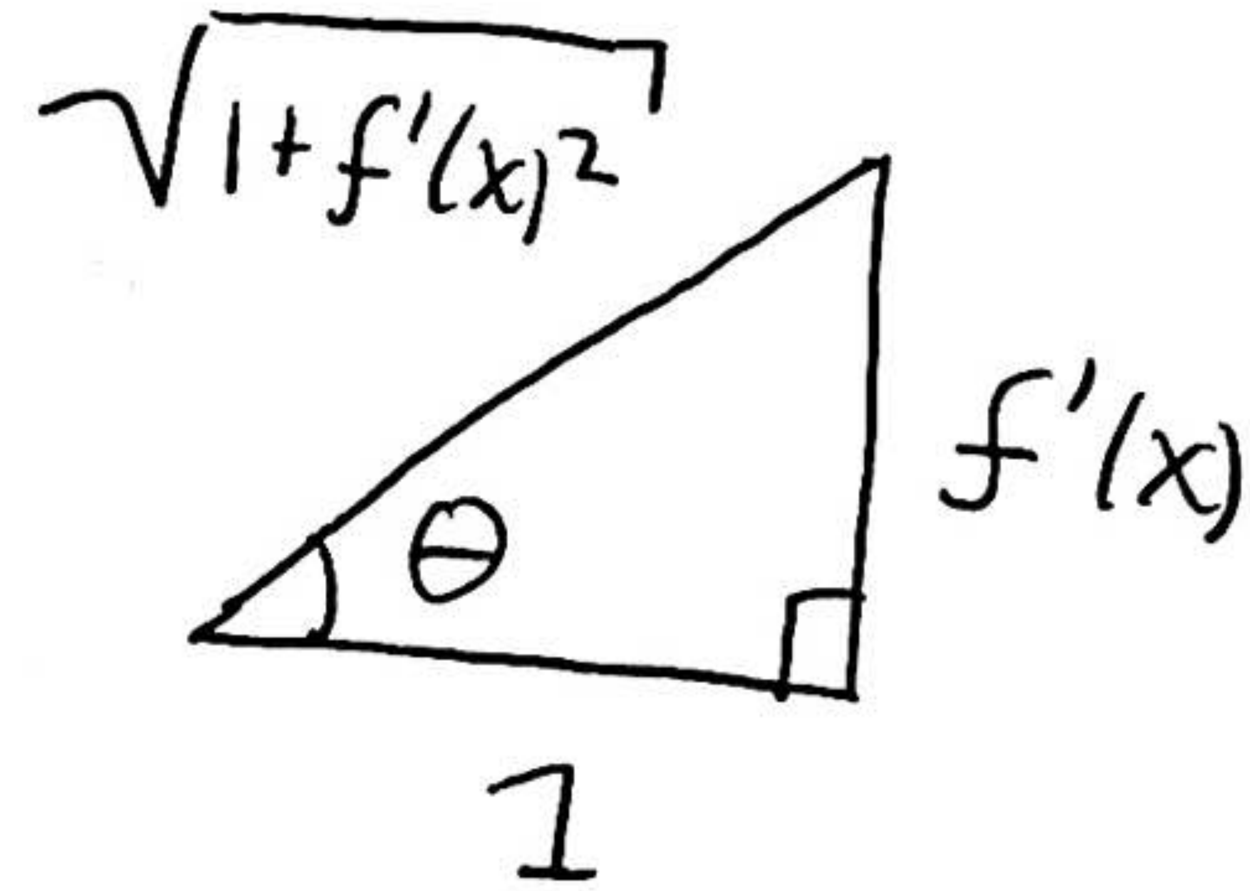
$$\alpha = \tan^{-1}(a)$$

$$\beta = \tan^{-1}(b)$$



$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx = (*)$$

$$\theta = \tan^{-1}(f'(x))$$



$$\tan\theta = f'(x)$$

$$u = f'(x)$$

$$du = f''(x)\, dx$$

$$\sec^2\theta\, d\theta = f''(x)\, dx$$

$$(*) = \int_{x=a}^{x=b} \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta$$

$$= \int_{x=a}^{x=b} \sec^3\theta\, d\theta$$

$$= \frac{1}{2}\Big[\ln|\tan\theta + \sec\theta| + \sec\theta\tan\theta\Big]\Big|_{x=a}^{x=b}$$

$$= (**)$$

$$\theta(b) = \tan^{-1}(f'(b))$$
$$\theta(a) = \tan^{-1}(f'(a))$$

$$\tan(\theta(b)) = f'(b)$$
$$\tan(\theta(a)) = f'(a)$$

$$\sec(\theta(b)) = \sqrt{f'(b)^2+1}$$
$$\sec(\theta(a)) = \sqrt{f'(a)^2+1}$$

$$(**) =$$
$$\frac{1}{2}\left[\ln\left|f'(b)+\sqrt{f'(b)^2+1}\right| + f'(b)\sqrt{f'(b)^2+1}\right]$$
$$-\frac{1}{2}\left[\ln\left|f'(a)+\sqrt{f'(a)^2+1}\right| + f'(a)\sqrt{f'(a)^2+1}\right]$$

$\boxed{\S 8.1}$

E4

$$y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$$

$$1 + (y')^2 = 1 + \left[\frac{1-2x}{2\sqrt{x}\sqrt{1-x}} + \frac{1}{2\sqrt{x}\sqrt{1-x}}\right]^2$$

$$= 1 + \frac{4(1-x)^2}{4x(1-x)}$$

$$= 1 + \frac{1-x}{x}$$

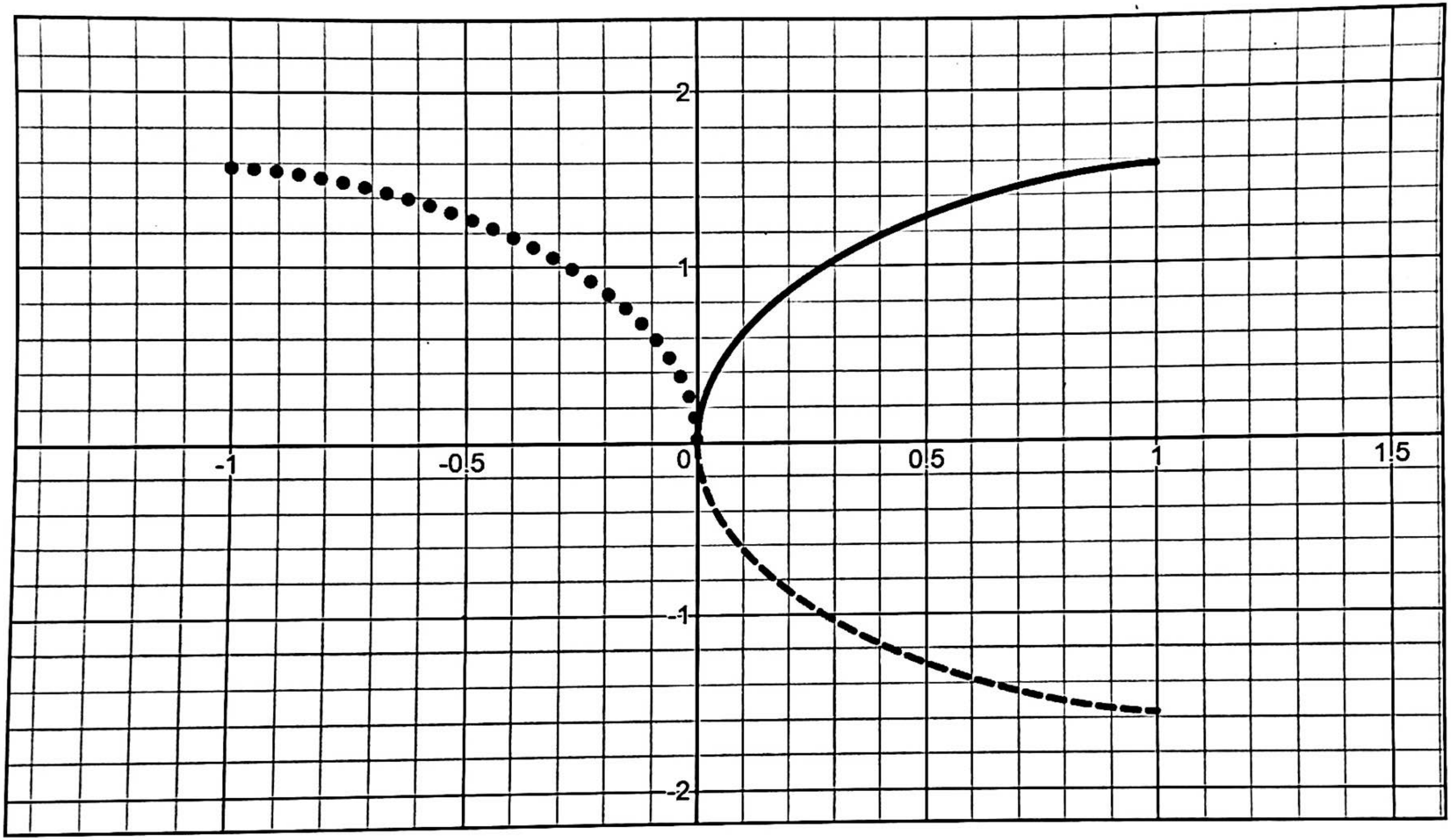
$$= \frac{x+1-x}{x}$$

$$= \frac{1}{x}$$

$$L = \int_0^1 \sqrt{\frac{1}{x}}\, dx$$

$$= \int_0^1 x^{-\frac{1}{2}} = 2x^{\frac{1}{2}}\Big|_0^1$$

$$= 2.$$



$f(x) = y = \sqrt{x-x^2} + \sin^{-1}(-\sqrt{x})$

Dom $f = [0,1]$

$$x = y^2$$

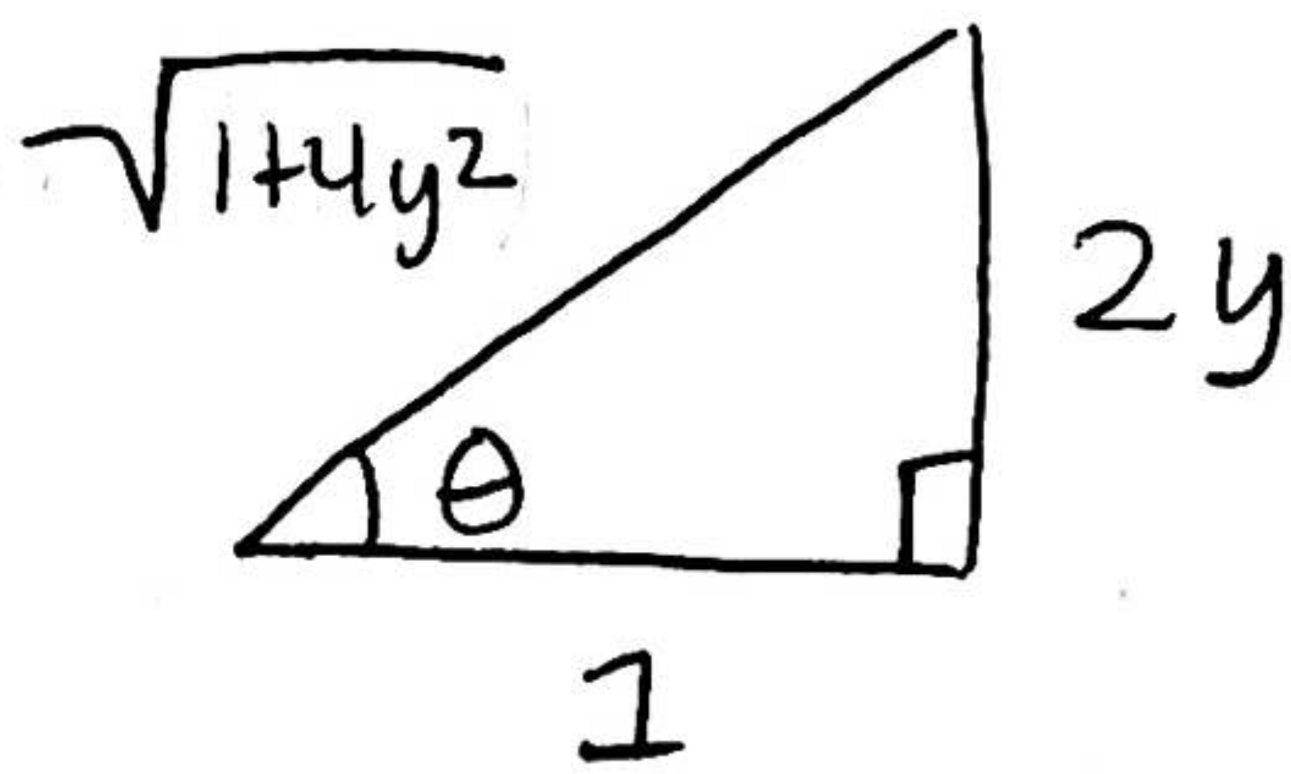
$$\frac{dx}{dy} = 2y$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + 4y^2$$

$$L = \int_0^1 \sqrt{1+4y^2}\, dy$$

$$y = \frac{1}{2}\tan\theta \qquad \frac{2y}{1} = \tan\theta$$

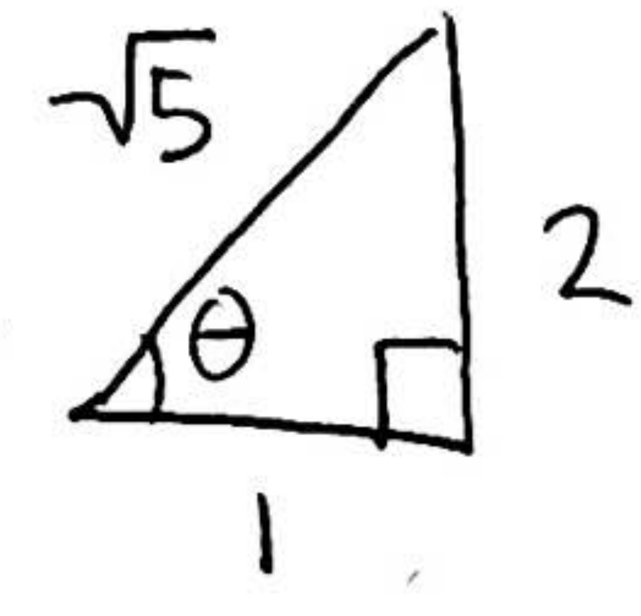
$$dy = \frac{1}{2}\sec^2\theta\, d\theta$$



$$0 = \frac{1}{2}\tan\theta \qquad 1 = \frac{1}{2}\tan\theta$$

$$\therefore\ \theta = 0 \qquad 2 = \tan\theta$$

$$\tan^{-1}(2) = \theta$$



$$L = \int_0^{\tan^{-1}(2)} \frac{1}{2}\sec^2\theta \sqrt{1+4\left(\tfrac{1}{2}\tan\theta\right)^2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^2\theta \sqrt{1+\tan^2\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta \, d\theta = (*)$$

$$\int \sec^3\theta \, d\theta$$

$u = \sec\theta$ $\qquad dv = \sec^2\theta \, d\theta$

$du = \sec\theta\tan\theta \, d\theta$ $\qquad v = \tan\theta$

$$uv - \int v\, du$$

$$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \, d\theta$$

$$I = \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta \, d\theta$$

$$2I = \sec\theta\tan\theta + \int \sec\theta \, d\theta$$

$$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$

$$I = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

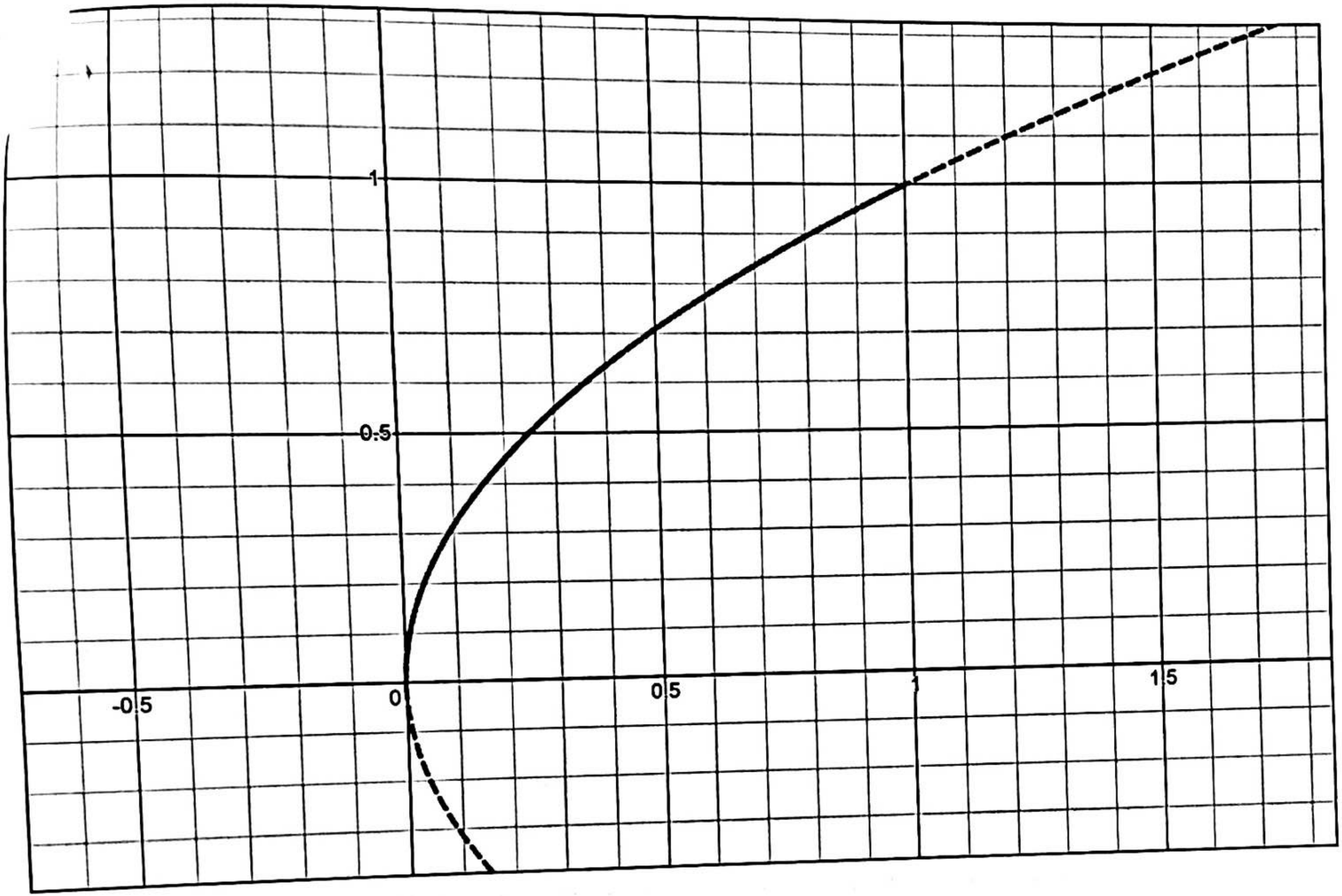
$$(*) = \frac{1}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_0^{\tan^{-1}(2}$$

$$= \frac{1}{4}\Big[(\sqrt{5})[2] + \ln|(\sqrt{5}) + 2|$$

$$- (1)(0) - \ln|1+0|\Big]$$

$$= \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2)$$

$$\approx 1.5$$


1
0.5
-0.5
0
0.5
1
1.5

FIND THE LENGTH OF $f(x) = \sqrt{x^3}$ BETWEEN $x = 0$ & $x = 4$.

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$= \frac{4}{9}\int_1^{\sqrt{10}} \sqrt{u}\, du$$

$$= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{10}$$

$$= \frac{8}{27}\left(10\sqrt{10} - 1\right)$$

$$f(x) = x^{\frac{3}{2}}$$

$$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$$

$$\left[f'(x)\right]^2 = \frac{9}{4}x$$

$$u = 1 + \frac{9}{4}x$$

$$du = \frac{9}{4}dx$$

$$u(0) = 1$$

$$u(4) = 10$$

$\boxed{\text{EXAMPLE 1}}$

$\boxed{\S 8.1}$

FIND L FOR $y = \ln(x + \sqrt{x^2-1})$ ON $[1, \sqrt{2}]$ EXAMPLE 2

ASIDE

$$y = \ln(x + \sqrt{x^2-1})$$

$$e^y = x + \sqrt{x^2-1}$$

$$y' e^y = 1 + \frac{1}{2}(2x)(x^2-1)^{-\frac{1}{2}}$$

$$y' e^y = 1 + x(x^2-1)^{-\frac{1}{2}}$$

$$y' = \frac{1 + \frac{x}{\sqrt{x^2-1}}}{x + \sqrt{x^2-1}}$$

$$= \frac{\sqrt{x^2-1} + x}{(x+\sqrt{x^2-1})} \cdot \frac{1}{\sqrt{x^2-1}}$$

$$1 + (y')^2 = 1 + \frac{1}{x^2-1}$$

$$= \frac{x^2-1+1}{x^2-1}$$

$$= \frac{x^2}{x^2-1}$$

$$L = \int_a^b \sqrt{1+f'(x)}\, dx$$

$$u = x^2+1$$

$$du = 2x\, dx$$

$$u(1) = 2$$

$$u(\sqrt{2}) = 3$$

$$L = \int_1^{\sqrt{2}} \sqrt{\frac{x^2}{x^2+1}}\, dx$$

$$= \int_1^{\sqrt{2}} \frac{x}{\sqrt{x^2+1}}\, dx$$

$$= \frac{1}{2}\int_{x=1}^{x=\sqrt{2}} \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{2}\int_2^3 u^{-\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_2^3$$

$$= \frac{1}{2}\cdot 2\left(\sqrt{3}-\sqrt{2}\right)$$

$$= \sqrt{3}-\sqrt{2}$$

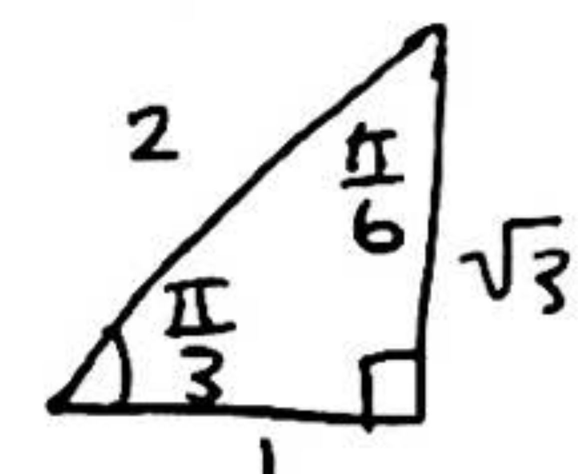
$\boxed{E2}$

§8.1

$y = x^2$

$y' = 2x$

$1 + (y')^2 = 1 + 4x^2$



$2x = \tan\theta$

$4x^2 = \tan^2\theta$

$2\,dx = \sec^2\theta \, d\theta$

$$L = \int_0^{\frac{\sqrt{3}}{2}} \sqrt{1+4x^2}\, dx$$

$2(0) = \tan\theta_1$

$\theta_1 = 0$

$2\left(\frac{\sqrt{3}}{2}\right) = \tan\theta_2$

$\sqrt{3} = \tan\theta_2$

$\theta_2 = \frac{\pi}{3}$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2\theta}\, \sec^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left[\ln|\tan(x) + \sec(x)| + \sec(x)\tan(x)\right]$$

$$= \frac{1}{2}\left(\ln\left|\tan\left(\tfrac{\pi}{3}\right) + \sec\left(\tfrac{\pi}{3}\right)\right| + \sec\tfrac{\pi}{3}\tan\left(\tfrac{\pi}{3}\right) - \ln|\tan(0) + \sec(0)| + \sec(0)\tan(0)\right)$$

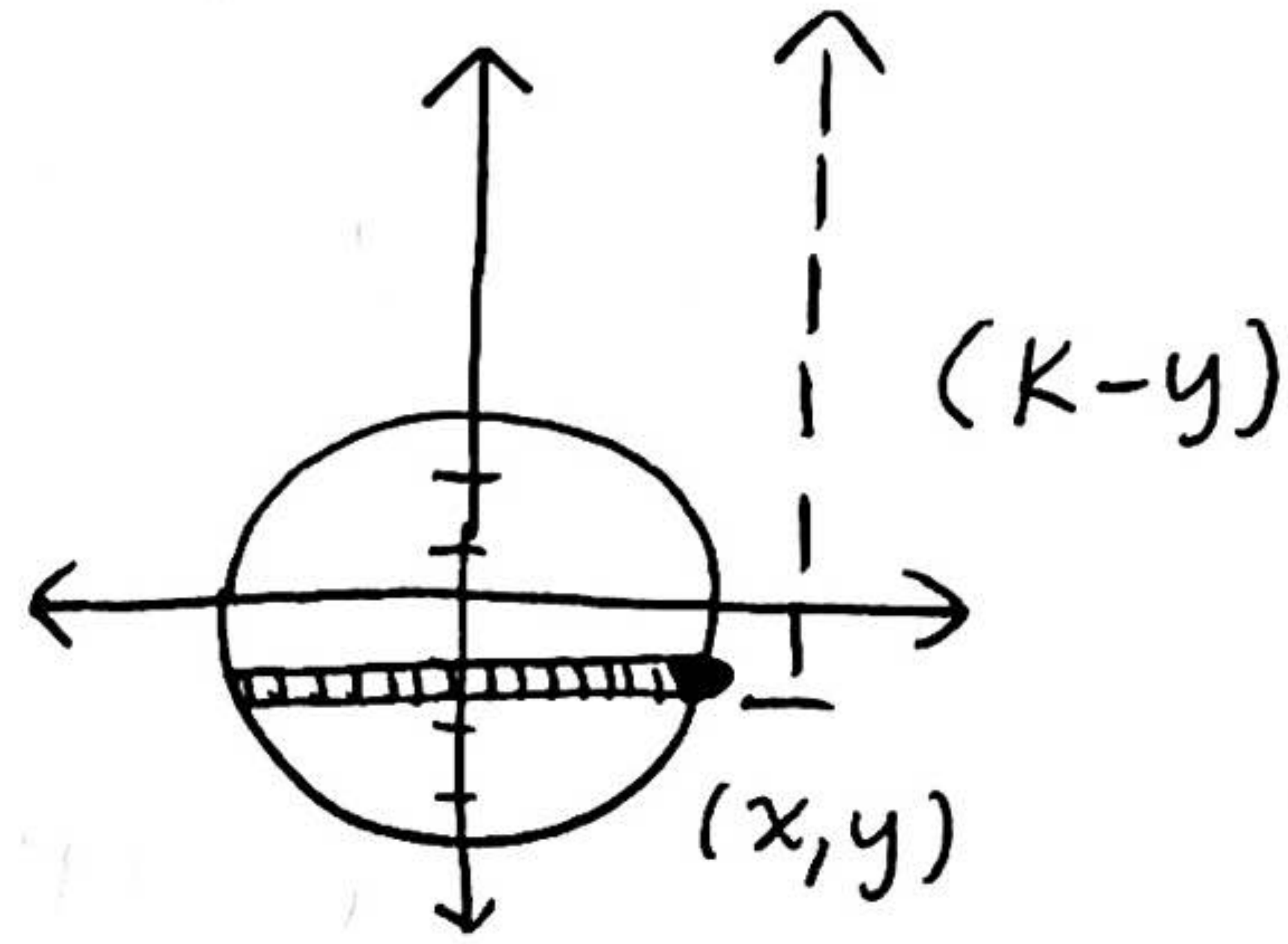
$$= \frac{1}{2}\left(\ln|\sqrt{3}+2| + 2\sqrt{3}\right)$$

$$\approx 2.39$$

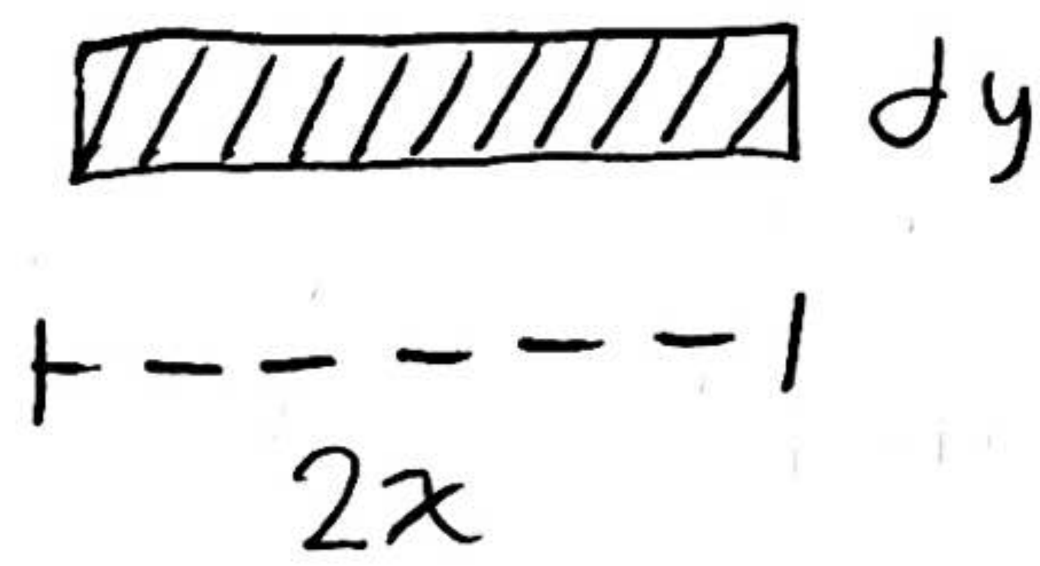
$$x^2 + y^2 = r^2$$

$$x = \pm\sqrt{r^2 - y^2}$$

$$2x = 2\sqrt{r^2 - y^2}$$



$$dF = ma$$
$$= \rho V a$$
$$= \rho g D A$$
$$= \rho g (k-y)(2x)\, dy$$
$$= \rho g (k-y)\, 2\sqrt{r^2-y^2}\, dy$$



$$F = \rho g \int_{-r}^{r} 2k\sqrt{r^2-y^2} - 2y\sqrt{r^2-y^2}\, dy$$

$$= 2\rho g \int_{-r}^{r} k\sqrt{r^2-y^2}\, dy$$

$$= 4\rho g \int_{0}^{r} k\sqrt{r^2-y^2}\, dy$$

$$= 4\rho g k \left[\frac{\pi r^2}{4}\right]$$

$$= \rho g k \pi r^2$$

#1

$m > k$

$k\pi r^2 + \frac{1}{2} > k$

$2k\pi r^2 + 1 > 2k$

$2k\pi r^2 > 2k - 1$

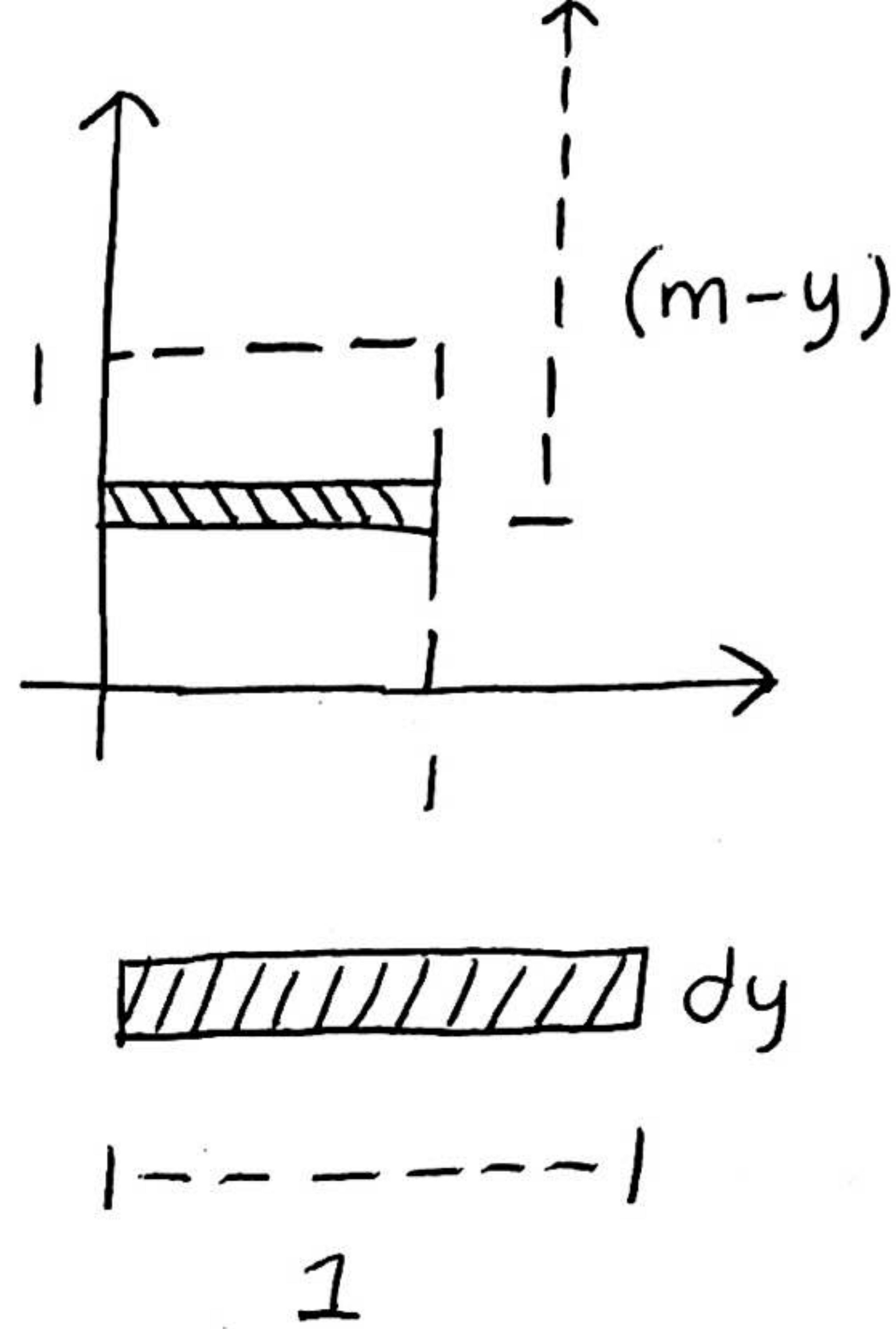
$$r^2 > \frac{2k-1}{2k\pi}$$

$$r > \sqrt{\frac{2k-1}{2k\pi}} > \sqrt{\frac{2k-2}{4k}}$$

$$= \sqrt{\frac{k-1}{2k}}$$

$\boxed{\#1}$

$$dF = ma$$
$$= \rho V a$$
$$= \rho g D A$$
$$= \rho g (m-y)\, dy$$



$$F = \int_0^1 \rho g (m-y)\, dy$$
$$= \rho g \left[my - \tfrac{1}{2} y^2 \right]_0^1$$
$$= \rho g \left(m - \tfrac{1}{2}\right)$$

$$F_s = F_c$$
$$\rho g \left(m - \tfrac{1}{2}\right) = \rho g k \pi r^2$$
$$m - \tfrac{1}{2} = k\pi r^2$$
$$m = k\pi r^2 + \tfrac{1}{2}$$

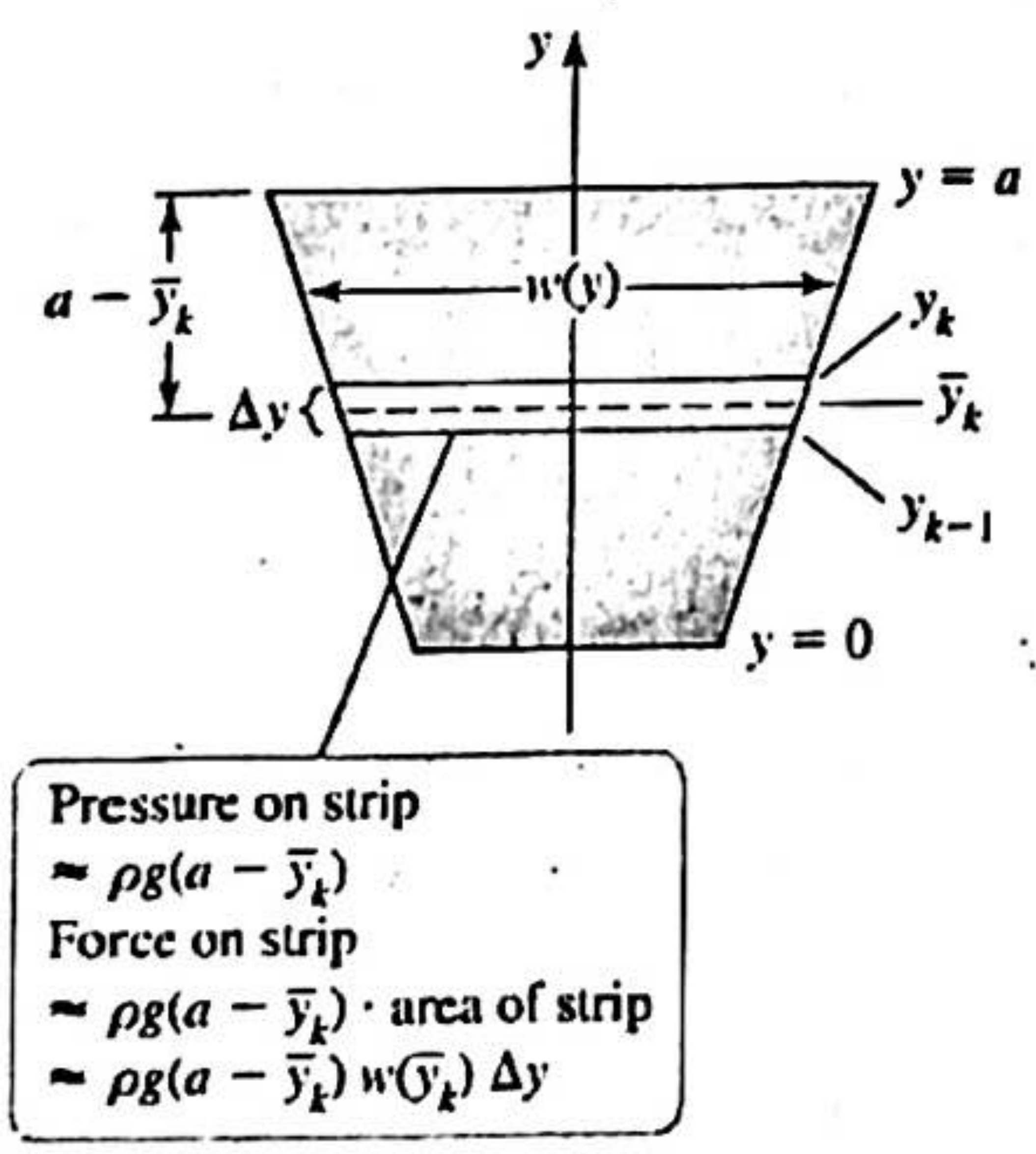
sps $m = 20$

$$20 > k\pi r^2 + \tfrac{1}{2}$$
$$40 > 2k\pi r^2 + 1$$
$$39 > 2k\pi r^2$$
$$\tfrac{39}{2} > k\pi r^2$$
$$\tfrac{39}{2} > k(3) r^2$$
$$\tfrac{13}{2} > k r^2$$

$\boxed{\#1}$

§8.3 EXAMPLE

A DIVING POOL THAT IS 4m DEEP & FULL OF WATER HAS A VIEWING WINDOW ON ONE OF IT'S WALLS. FIND THE FORCE ON A WINDOW THAT IS SQUARE, $\frac{1}{2}$m on a SIDE, W/ THE LOWER EDGE OF THE WINDOW 1M FROM THE BOTTOM OF THE POOL.



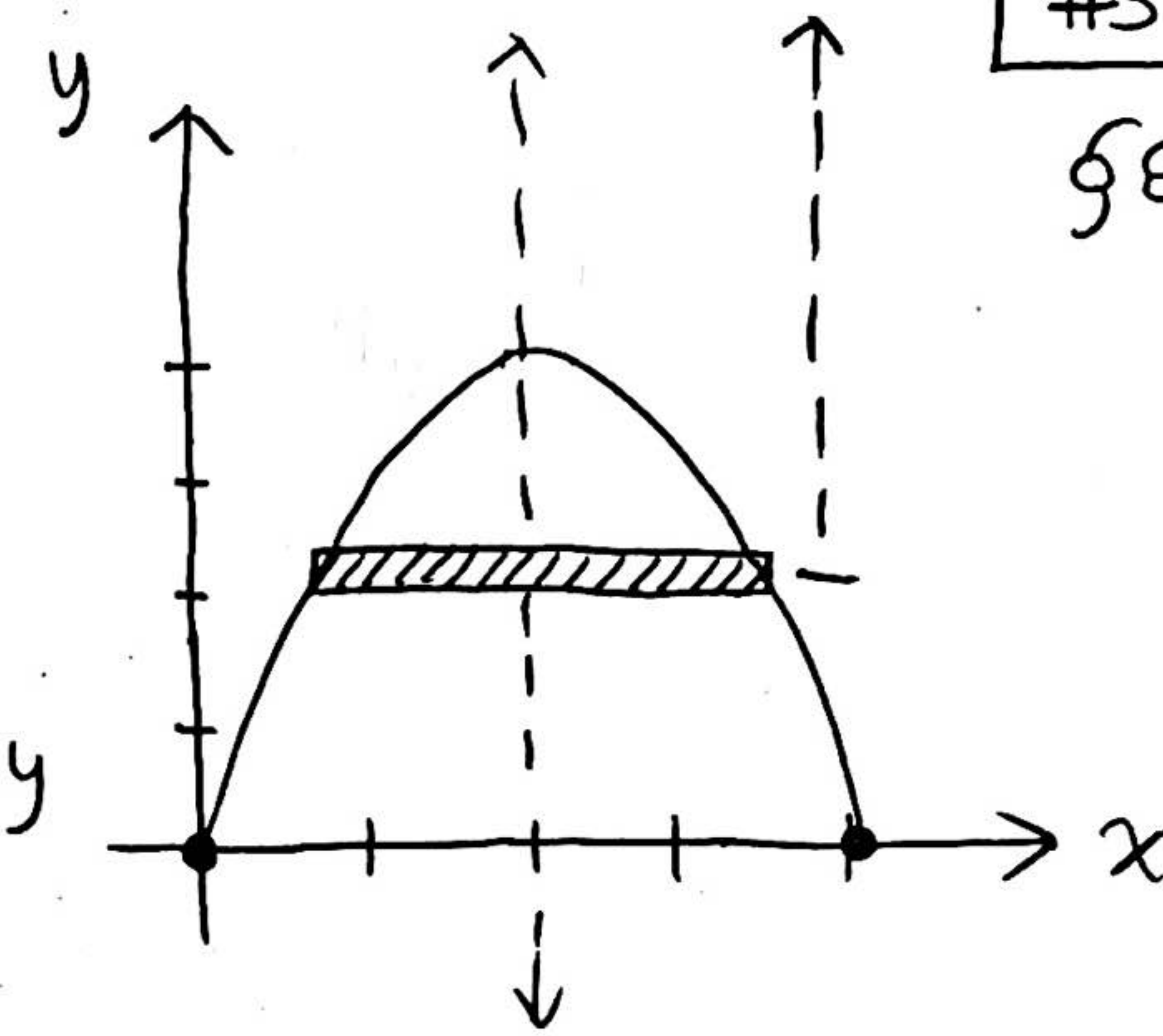
$$F_k = \underbrace{w(\bar{y}_k)\,\Delta y}_{\text{area of strip}}\,\underbrace{\rho g(a - \bar{y}_k)}_{\text{pressure}}.$$

$$F \approx \sum_{k=1}^{n} F_k = \sum_{k=1}^{n} \rho g(a - \bar{y}_k) w(\bar{y}_k)\,\Delta y.$$

$$F = \lim_{n\to\infty} \sum_{k=1}^{n} \rho g(a - \bar{y}_k) w(\bar{y}_k)\,\Delta y = \int_0^a \rho g(a - y) w(y)\,dy.$$

$\boxed{\#3}$

§8.3



$$dF = \rho g(10-y)(2x)\,dy$$

$$= \rho g(10-y)(2\sqrt{4-y})\,dy$$

$$F = \int_0^{10} 2\rho g(10-y)\sqrt{4-y}\,dy$$

$$P = \int_0^{10} \rho g(10-y)\,dy$$

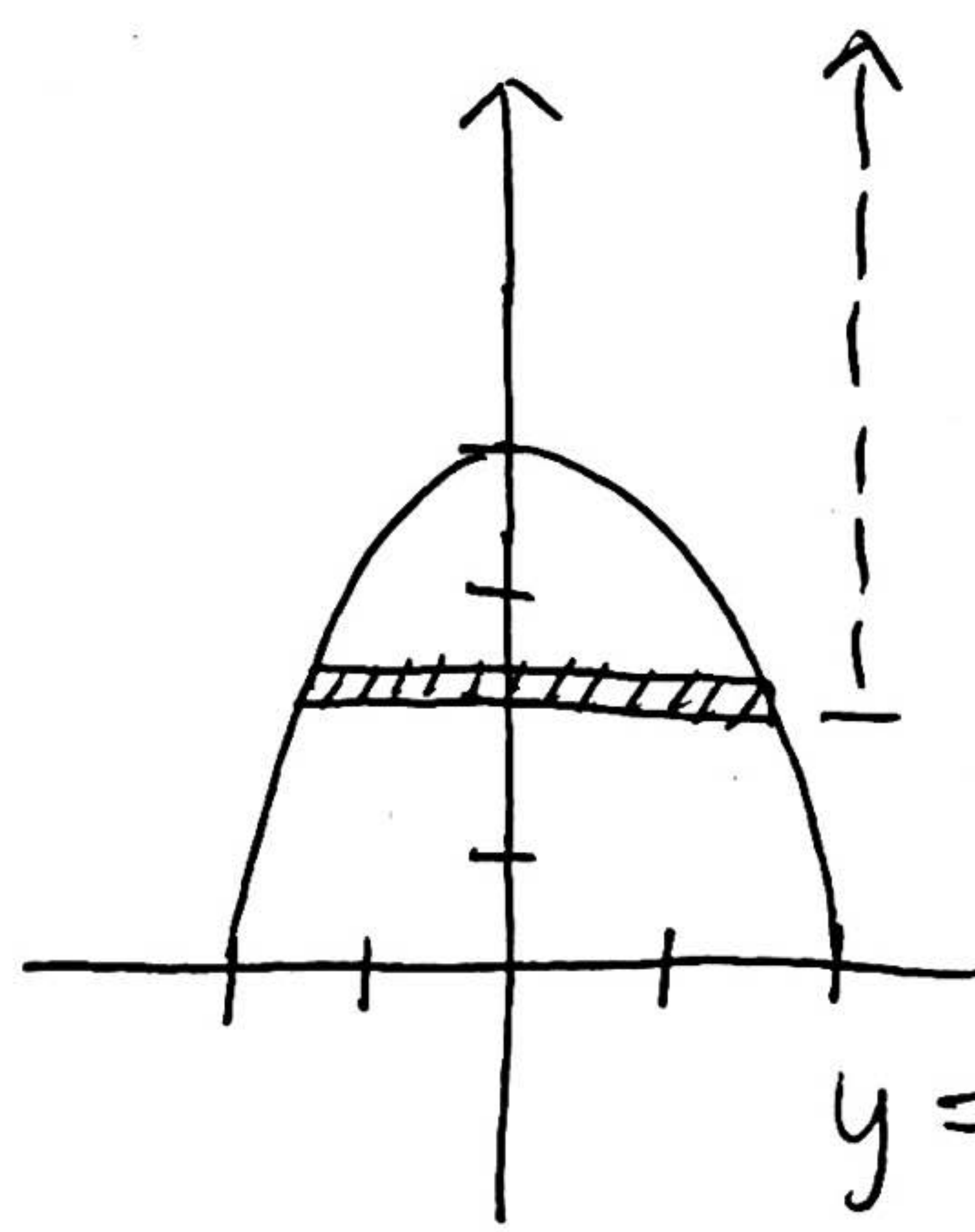
$$y = a(x)(4-x)$$

$$4 = a(2)(2)$$

$$4 = 4a \quad \therefore a = 1$$

$$y = 4x - x^2$$

$\boxed{\text{OR}}$



$$y = 4 - x^2$$

$$x = \pm\sqrt{4-y}$$

A VERTICAL DAM HAS A PARABOLIC GATE AS SHOWN IN THE DIAGRAM. FIND THE HYDROSTATIC FORCE & PRESSURE ON THE GATE

$$P = \frac{F}{A}$$

$$F = PA$$

$$V = Ah$$

$$A = \frac{V}{h}$$

$$P = F\frac{1}{A}$$

$$P = F\frac{h}{V}$$

$$= ma\frac{h}{V}$$

$$= \rho V a\frac{h}{V}$$

$$= \rho g h$$

$$F = PA$$

$$= \rho g(10-y_i)(2x_i)\Delta y$$

$$= \rho g(10-y)(2x)\,dy$$

$$= \rho g(10-y)(2\sqrt{4-y})\,dy$$

$$u = \sqrt{4-y}$$

$$u^2 = 4-y$$

$$u^2+6 = 10-y$$

$$2u\,du = -dy$$

$$u(0) = 2$$

$$u(4) = 0$$

$$F = \int_0^4 \rho g(10-y)(2\sqrt{4-y})\,dy$$

$$= 2(9800)\int_0^4 (10-y)\sqrt{4-y}\,dy$$

$$= 3.92(10^4)$$

$$= \int_{0}^{2} (u^2+6)(u^2)(u)\,du$$

? ?

$$= C\int_0^2 u^5+6u^3\,du$$

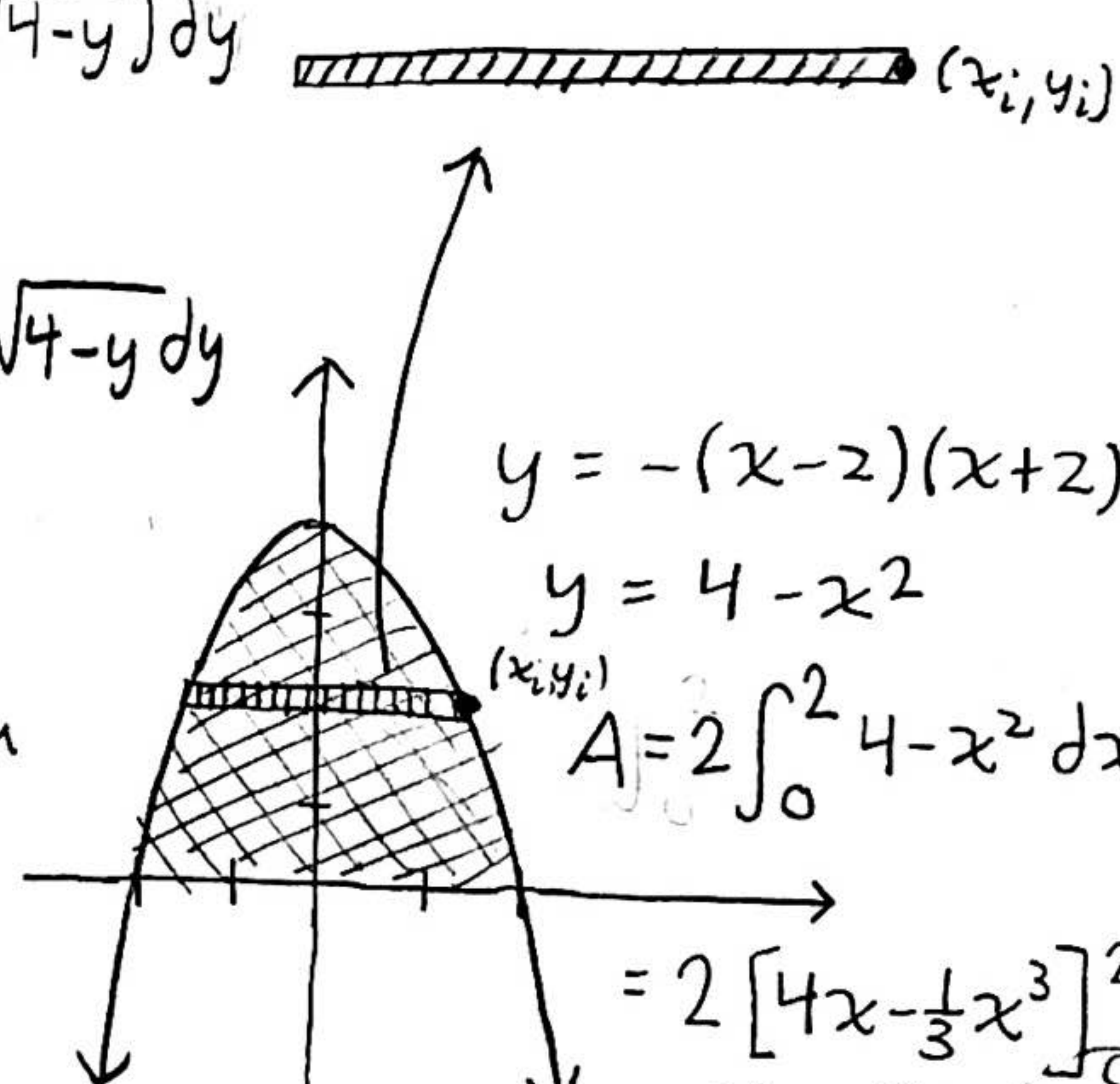
$$= C\left[\frac{1}{6}u^6+\frac{3}{2}u^4\right]_0^2$$

$$= 3.92(10^4)\left[\frac{64}{6}+3(8)\right]$$

$$= 3.92(10^4)[\text{NOISE}]$$

$$= \frac{10192}{75}\cdot 10^4 = \frac{101920000}{75}$$

$$= \boxed{\frac{4076800}{3}\text{ N}}$$



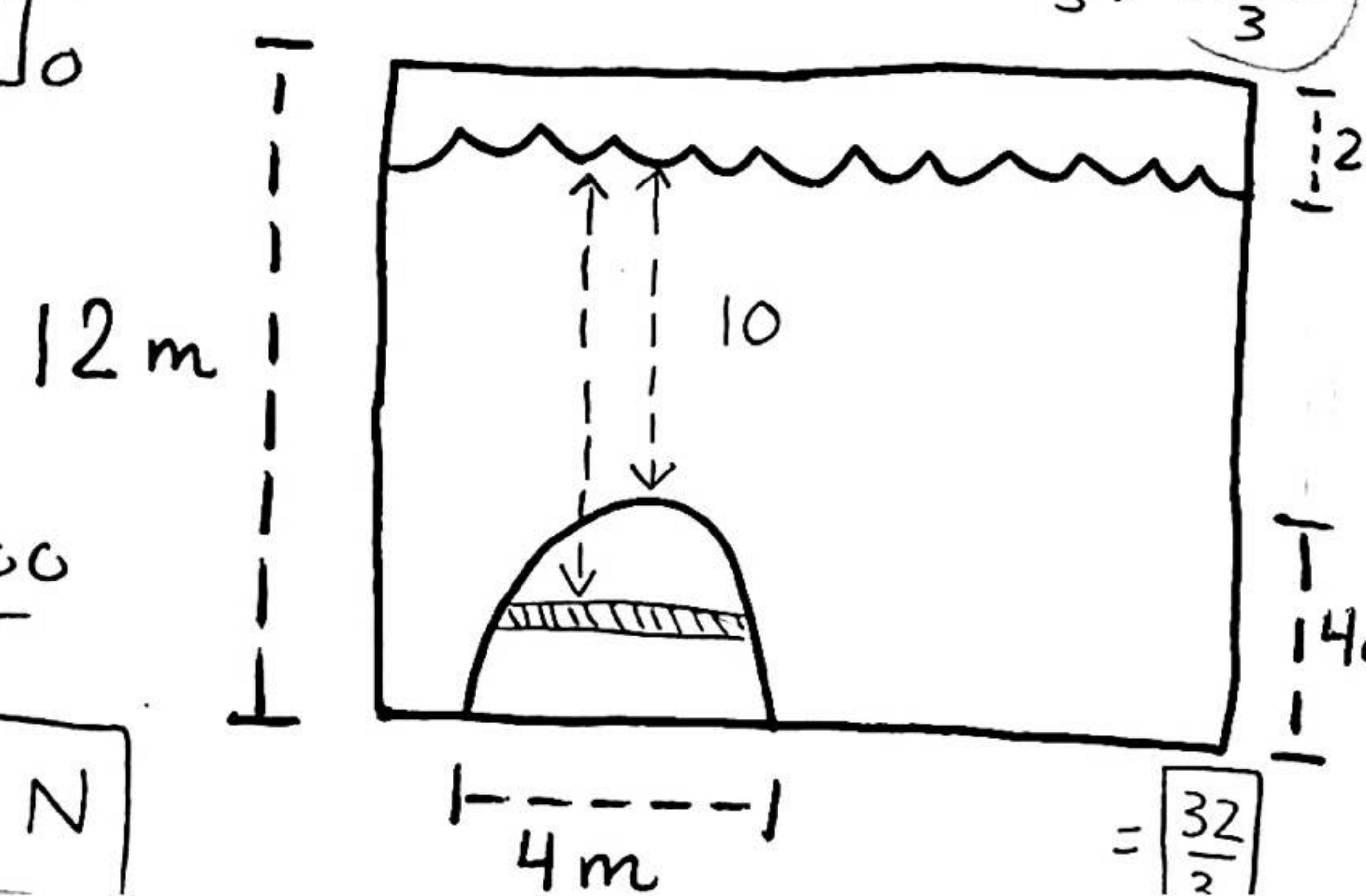
$$y = -(x-2)(x+2)$$

$$y = 4-x^2$$

$$A = 2\int_0^2 4-x^2\,dx$$

$$= 2\left[4x-\frac{1}{3}x^3\right]_0^2$$

$$= 2\left(8-\frac{8}{3}\right) = \frac{48-16}{3}$$



$$= \boxed{\frac{32}{3}}$$

/

A TANK W/ PARABOLIC FRONT
THAT IS SIX METERS & NINE METERS HIGH.
THE TANK EXTENDS BACK 10 METERS

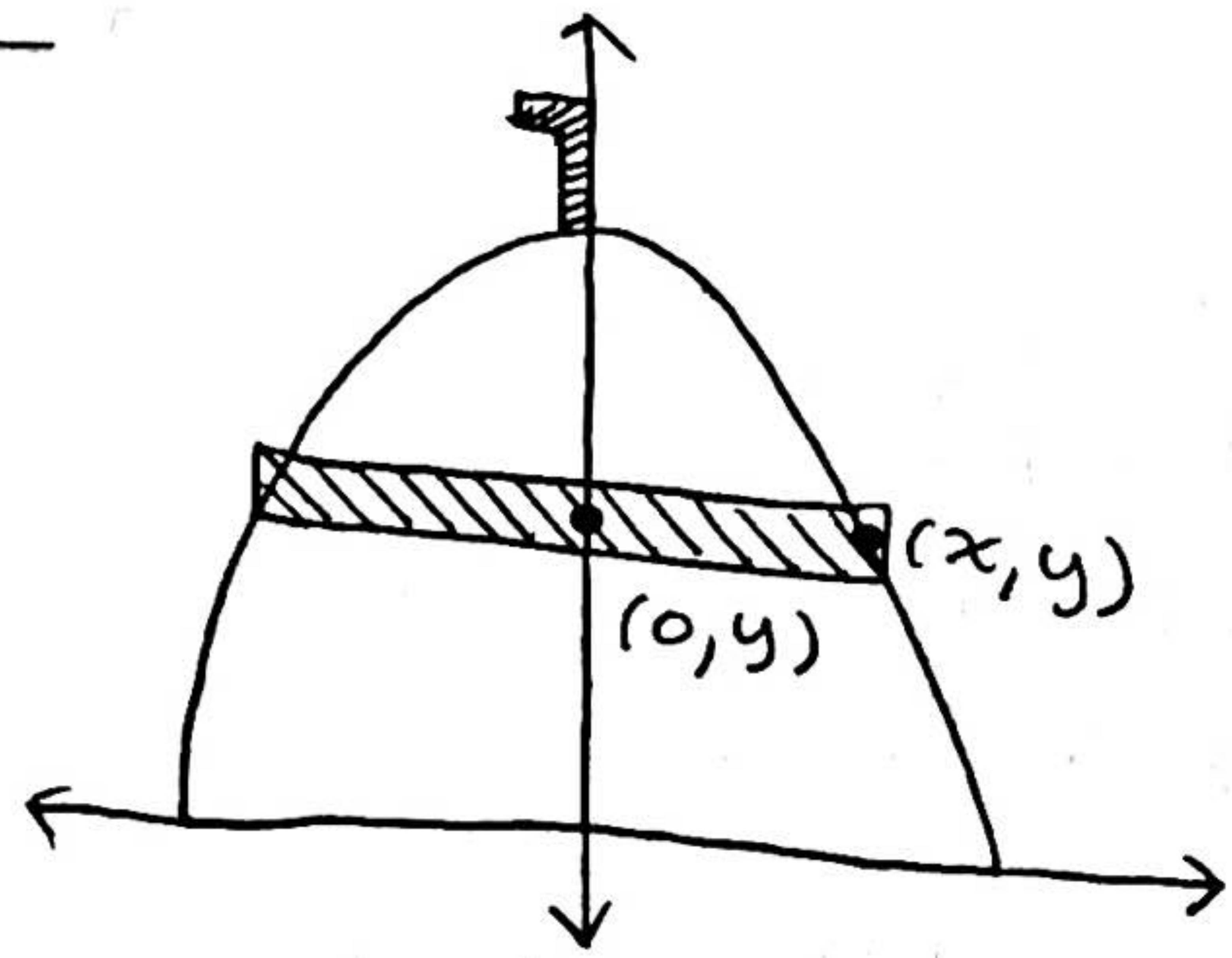
FIND THE VOLUME OF THE
TANK

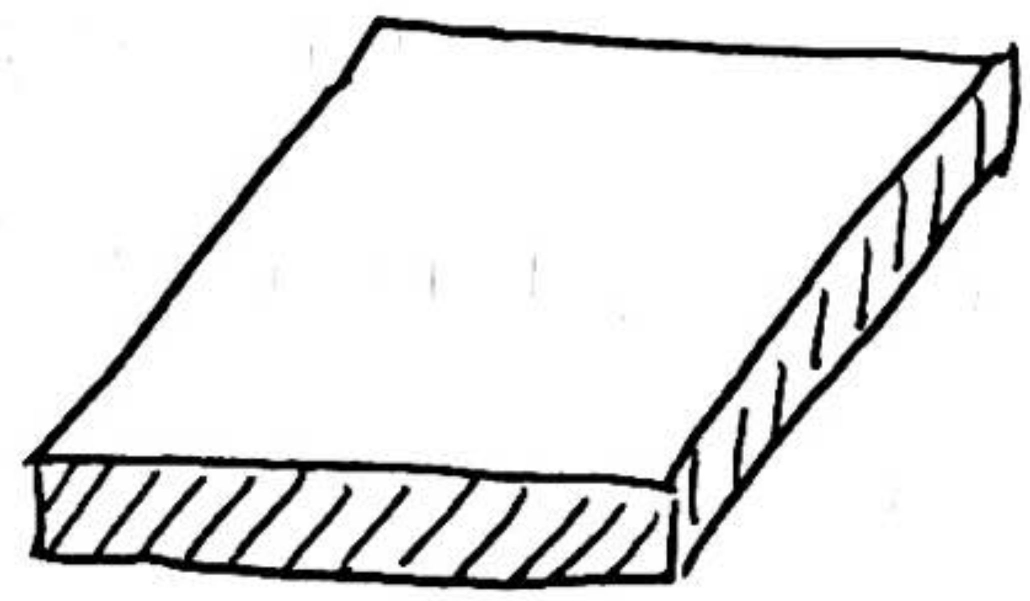
FIND THE HEIGHT OF THE WATER
IF HALF OF THE TANKS WATER IS
PUMPED OUT

FIND THE AMOUNT OF WORK TO
TO REMOVE HALF OF THE WATER

FIND THE HYDROSTATIC FORCE
ON THE BOTTOM OF THE TANK

CH 8
PROBLEM





$$V = lwh$$

$$dV = 2x(10)\,dy$$

$$dV = 20x\,dy$$

$$dV = 20\sqrt{9-y}\,dy$$

$$W = FD$$

$$= maD$$

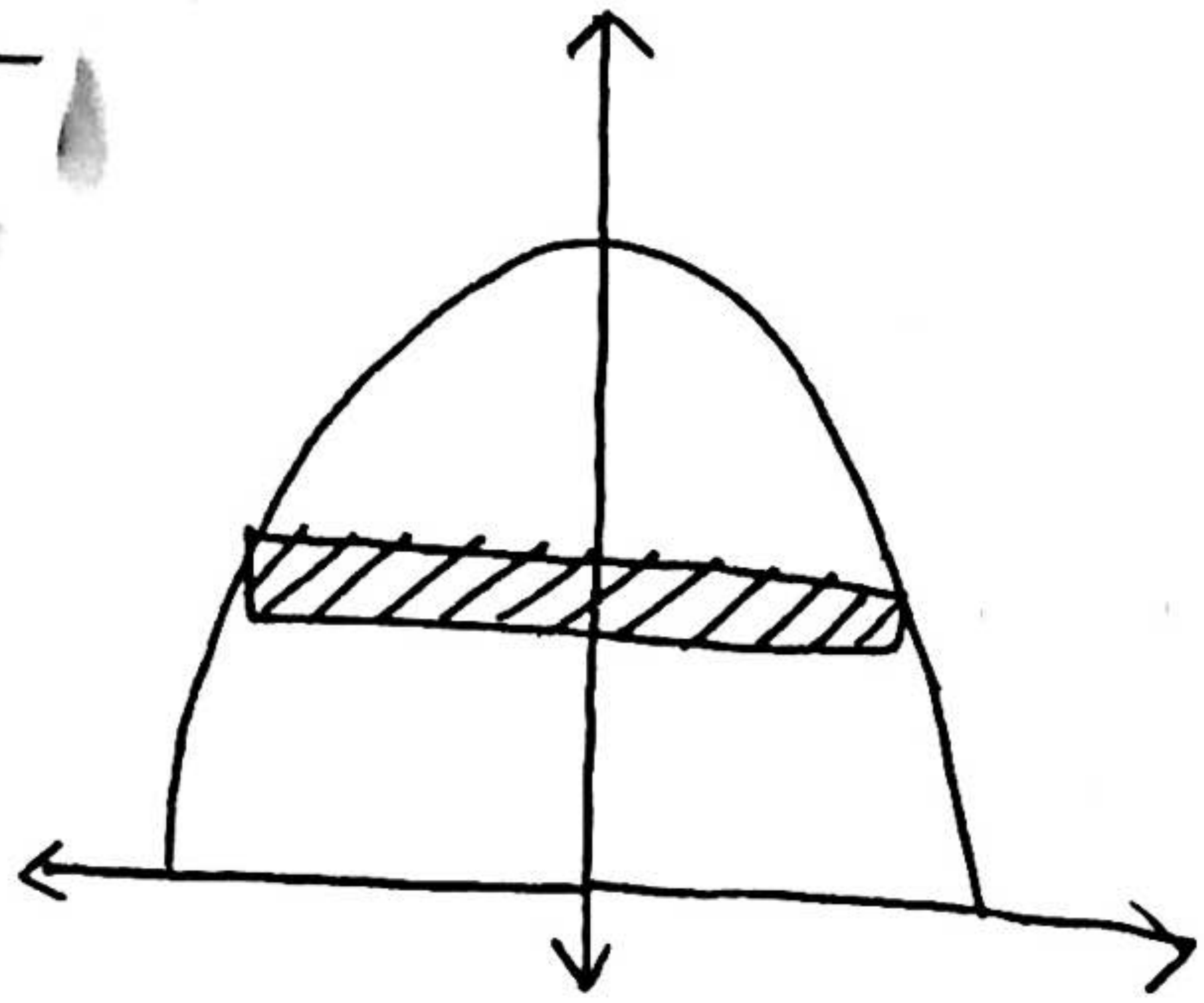
$$= \rho V g D$$

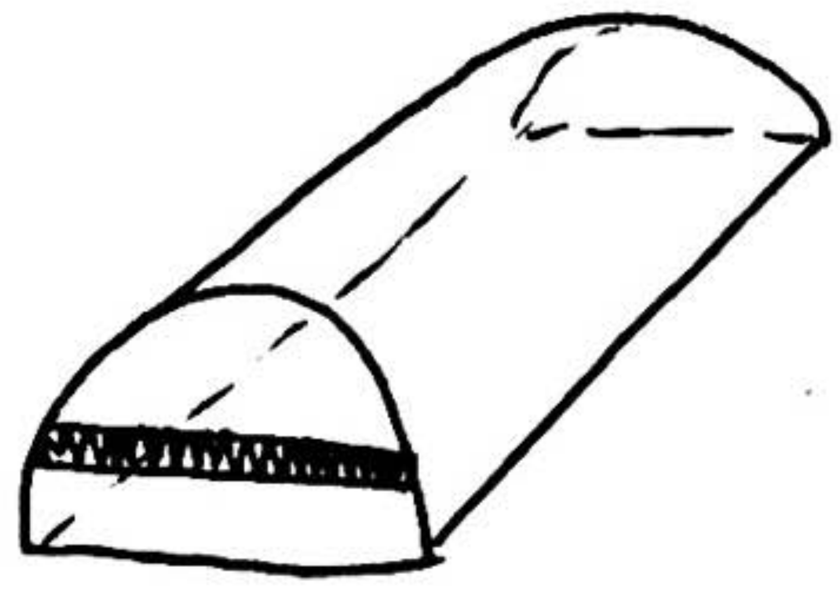
$$= \rho g V(10-y)$$

$$dW = \rho g(10-y)(20)\sqrt{9-y}\,dy$$

$$= 20\rho g(10-y)\sqrt{9-y}\,dy$$

$$W = 20\rho g\int_0^t (10-y)\sqrt{9-y}\,dy$$





$$V = l \cdot A$$

$$= (10)\int_{-3}^{3} 9 - x^2 \, dx$$

$$= 20 \int_{0}^{3} 9 - x^2 \, dx$$

$$= 20 \left[9x - \tfrac{1}{3}x^3 \right]_0^3$$

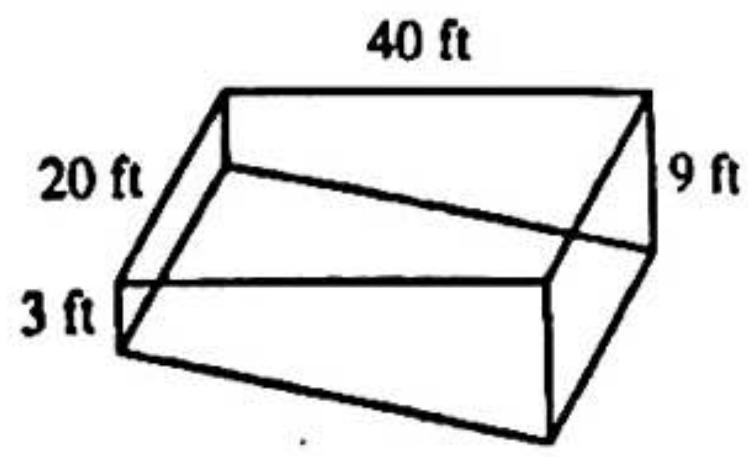
$$= 20 \left[27 - \frac{27}{3} \right]$$

$$= 20(27)\left(\frac{2}{3}\right) = 360 \quad m^3$$

[8.3.17]

17. A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, estimate the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.

Solution



17. (a) The area of a strip is $20\,\Delta x$ and the pressure on it is δx_i.

$$F = \int_0^3 \delta x 20\,dx = 20\delta\left[\tfrac{1}{2}x^2\right]_0^3 = 20\delta \cdot \tfrac{9}{2} = 90\delta$$
$$= 90(62.5) = 5625 \text{ lb} \approx 5.63 \times 10^3 \text{ lb}$$

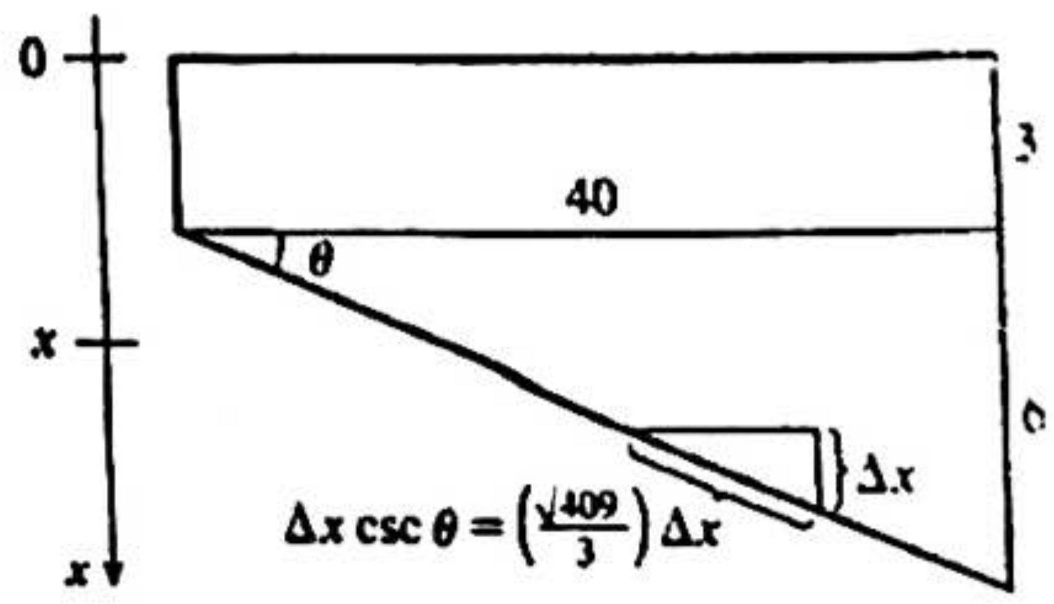
(b) $F = \int_0^9 \delta x 20\,dx = 20\delta\left[\tfrac{1}{2}x^2\right]_0^9 = 20\delta \cdot \tfrac{81}{2} = 810\delta = 810(62.5) = 50{,}625 \text{ lb} \approx 5.06 \times 10^4 \text{ lb.}$

(c) For the first 3 ft, the length of the side is constant at 40 ft. For $3 < x \le 9$, we can use similar triangles to find the length a:

$$\frac{a}{40} = \frac{9-x}{6} \quad\Rightarrow\quad a = 40 \cdot \frac{9-x}{6}.$$

$$F = \int_0^3 \delta x 40\,dx + \int_3^9 \delta x(40)\,\frac{9-x}{6}\,dx = 40\delta\left[\tfrac{1}{2}x^2\right]_0^3 + \tfrac{20}{3}\delta\int_3^9 (9x - x^2)\,dx = 180\delta + \tfrac{20}{3}\delta\left[\tfrac{9}{2}x^2 - \tfrac{1}{3}x^3\right]_3^9$$
$$= 180\delta + \tfrac{20}{3}\delta\left[\left(\tfrac{729}{2} - 243\right) - \left(\tfrac{81}{2} - 9\right)\right] = 180\delta + 600\delta = 780\delta = 780(62.5) = 48{,}750 \text{ lb} \approx 4.88 \times 10^4 \text{ lb}$$

(d) For any right triangle with hypotenuse on the bottom,



$$\sin\theta = \frac{\Delta x}{\text{hypotenuse}} \quad\Rightarrow$$

$$\text{hypotenuse} = \Delta x \csc\theta = \Delta x\,\frac{\sqrt{40^2 + 6^2}}{6} = \frac{\sqrt{409}}{3}\,\Delta x.$$

$$F = \int_3^9 \delta x 20\,\tfrac{\sqrt{409}}{3}\,dx = \tfrac{1}{3}\left(20\sqrt{409}\right)\delta\left[\tfrac{1}{2}x^2\right]_3^9$$
$$= \tfrac{1}{3} \cdot 10\sqrt{409}\,\delta(81 - 9) \approx 303{,}356 \text{ lb} \approx 3.03 \times 10^5 \text{ lb}$$

$$F_1 = \int_6^9 \rho g (40)(9-y)\,dy$$

$$= 40\rho g \int_6^9 9-y\,dy$$

$$= 40\rho g \left[9y - \tfrac{1}{2}y^2\right]_6^9$$

$$= 40\rho g \left[9(9-6) - \tfrac{1}{2}(9^2-6^2)\right]$$

$$= 40\rho g \left[27 - \tfrac{1}{2}(81-36)\right]$$

$$= 40\rho g \left[27 - \tfrac{45}{2}\right]$$

$$= 180\rho g$$

$$F_2 = \frac{20}{3}\rho g \left[\tfrac{9}{2}y^2 - \tfrac{1}{3}y^3\right]\Big|_0^6$$

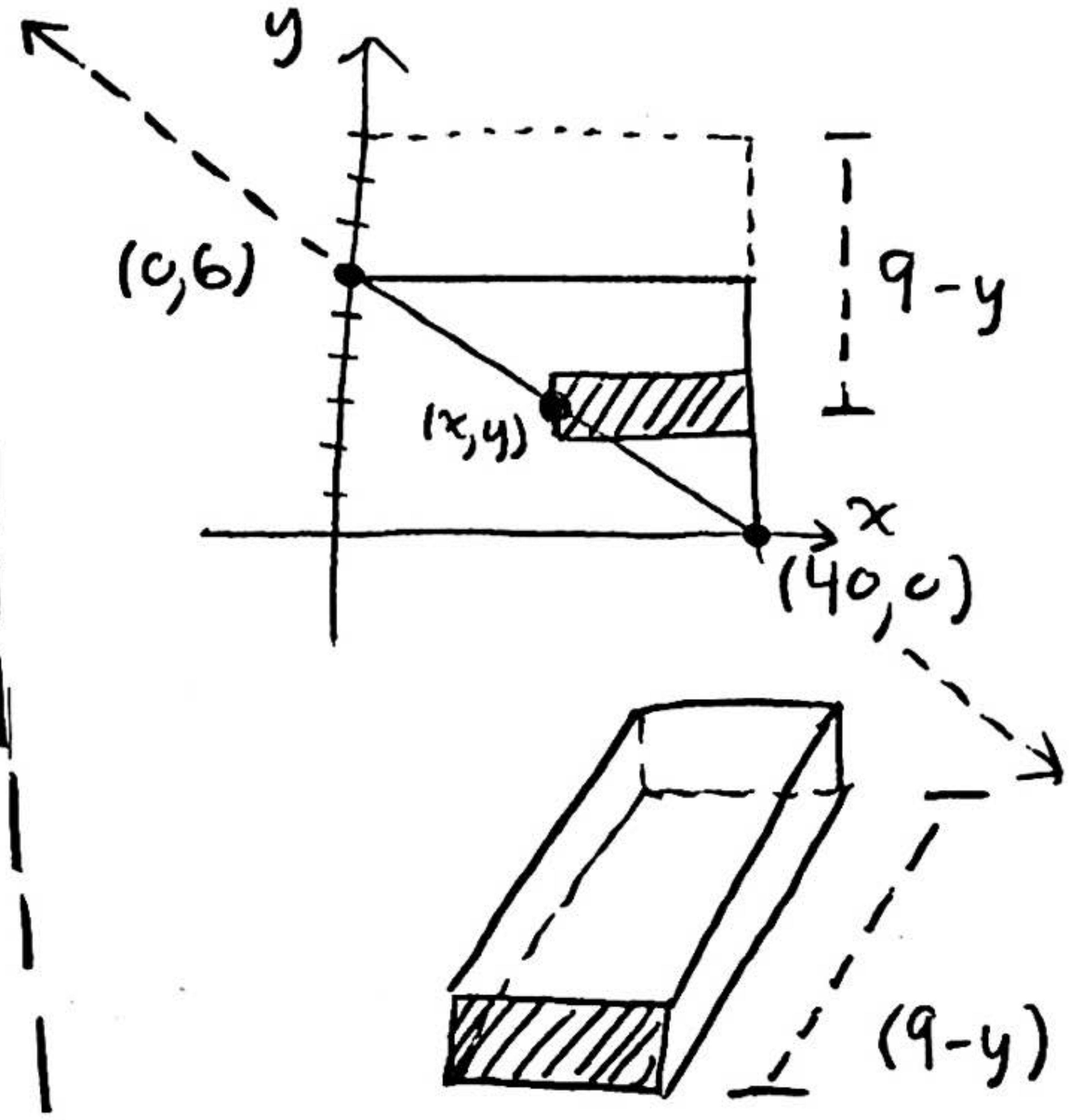
$$= 600\rho g$$

$$F_1 + F_2 = 780\rho g$$

$$= (780)(62.5)$$

$$\approx 4.88 \times 10^4$$

$$F = ma$$
$$= \rho V g$$
$$= \rho g A \cdot d$$
$$= \rho g (l \cdot w) d$$
$$= \rho g \left(\frac{120-20y}{3}\right)(9-y)\,dy$$



$$A = (40 - x)\,dy$$

$$F_2 = \rho g \int_0^6 \left[40 - \left[\frac{120-20y}{3}\right]\right](9-y)\,dy$$
$$= \rho g \int_0^6 \left(\frac{20}{3}y\right)(9-y)\,dy$$
$$= \frac{20}{3}\rho g \int_0^6 9y - y^2\,dy$$
$$= \frac{20}{3}\rho g \left[\frac{9}{2}y^2 - \frac{1}{3}y^3\right]\Big|_0^6$$

LINE

$(0,6)$ & $(40,0)$

$$m = \frac{0-6}{40-0} = \frac{-6}{40} = \frac{-3}{20}$$

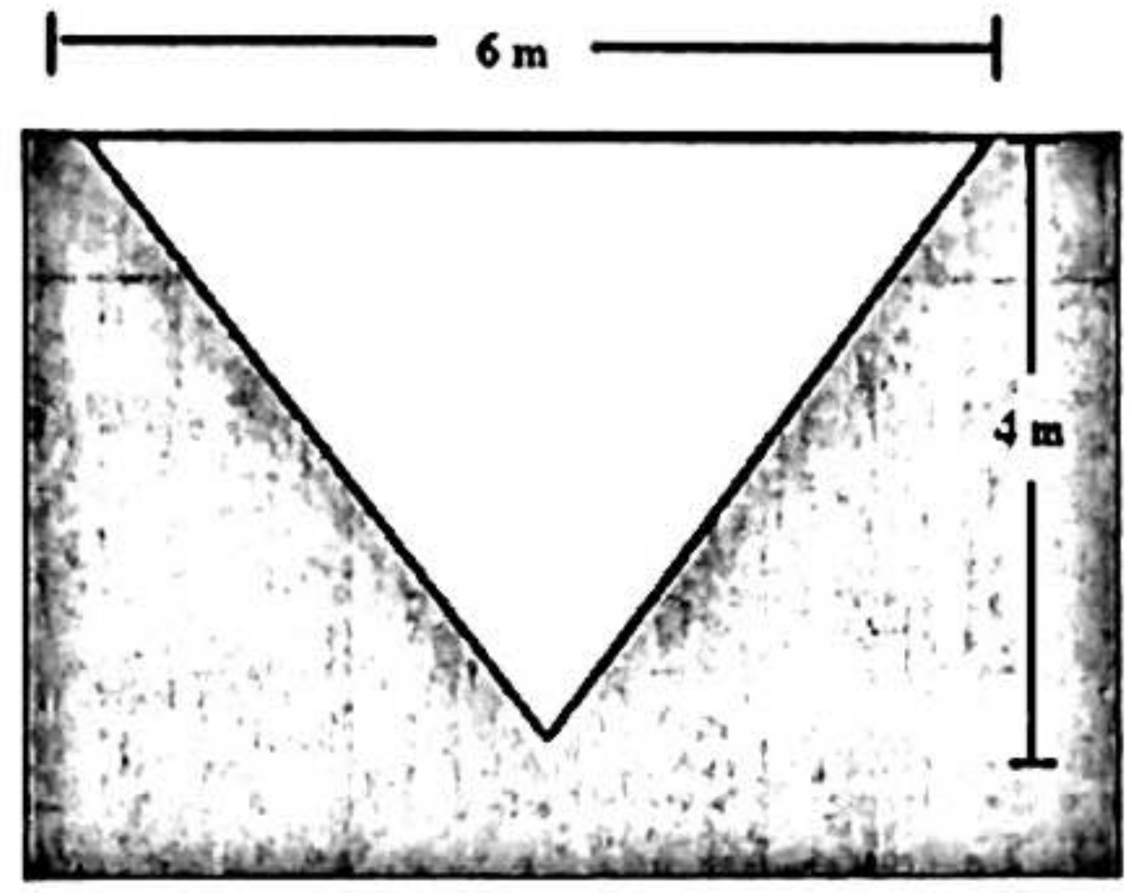
$$y = -\frac{3}{20}x + 6$$

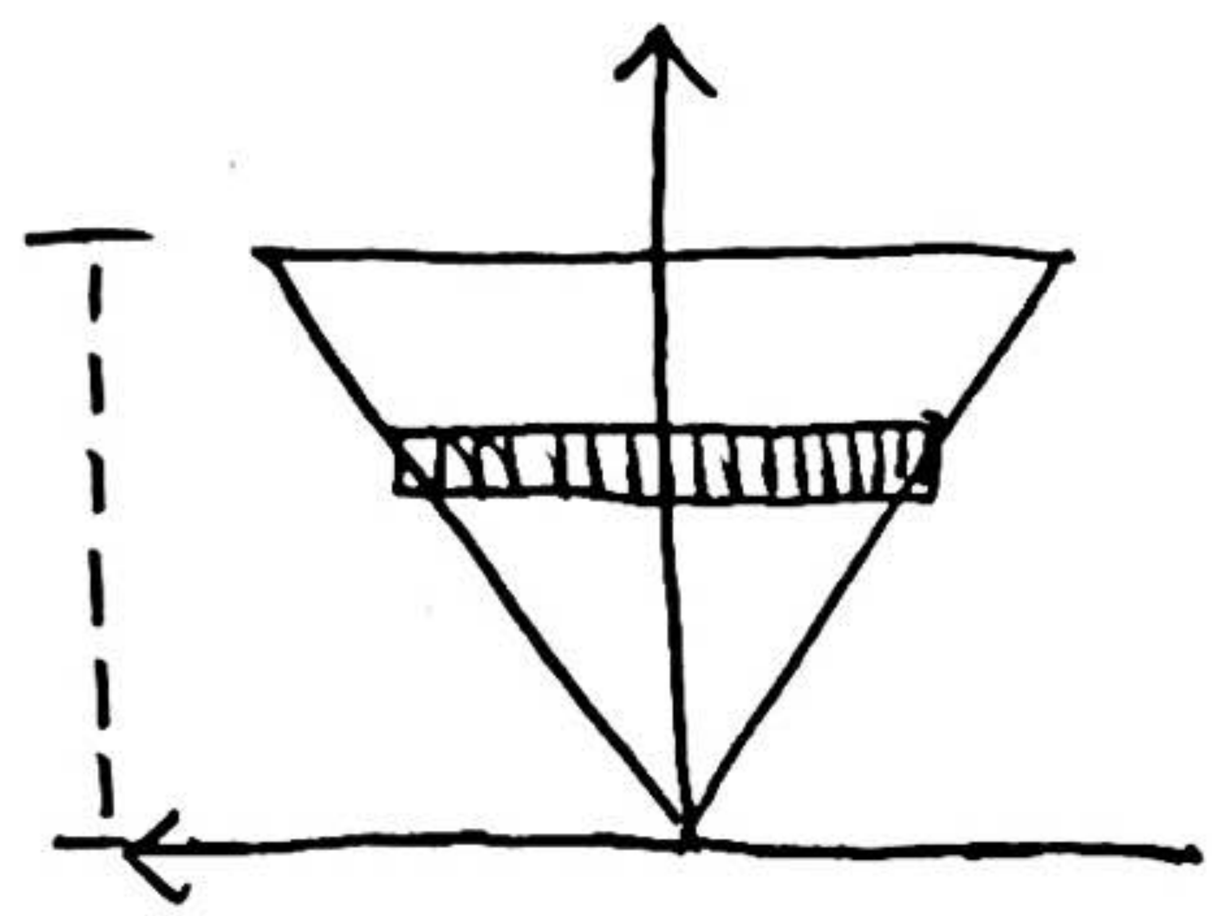
$$20y = -3x + 120$$

$$\frac{20y-120}{-3} = x$$

$$x = \frac{120-20y}{3}$$

8. Determine the hydrostatic force applied to the vertical plate below submerged in water.

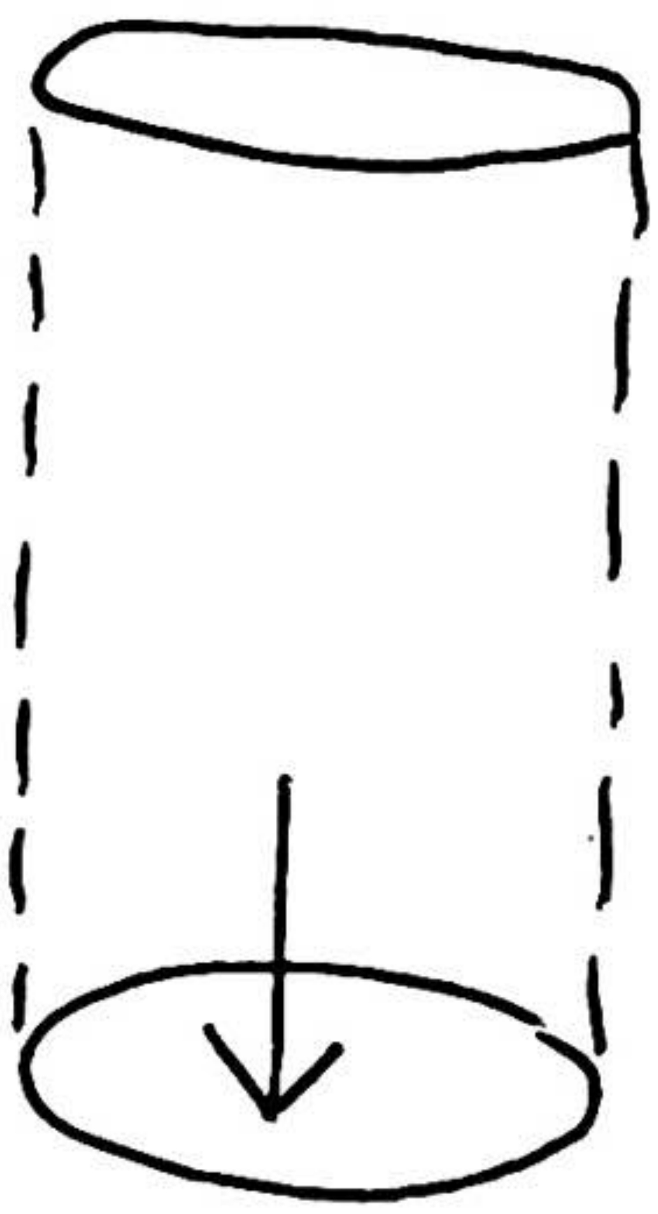


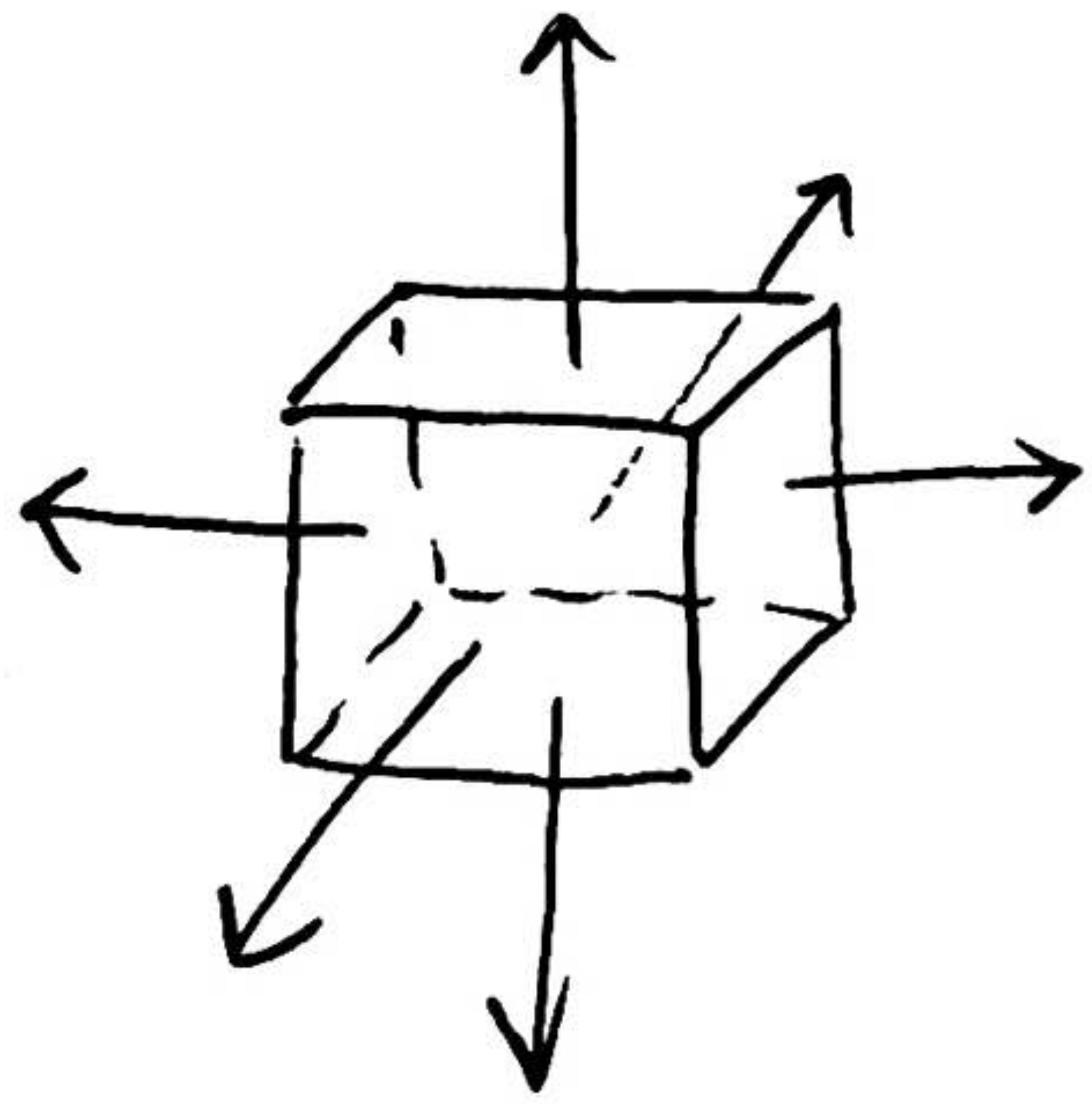


$F = P \cdot A$

§8.3

7. Determine the amount of pressure and force acting on the bottom of a cylindrical drum of olive oil with base radius of ½ m and height 1 m. (Note: the density of olive oil is $920\,{}^{kg}/_{m^3} \approx 57.4\,{}^{lb}/_{ft^3}$)





$F = ma$
$= \rho$

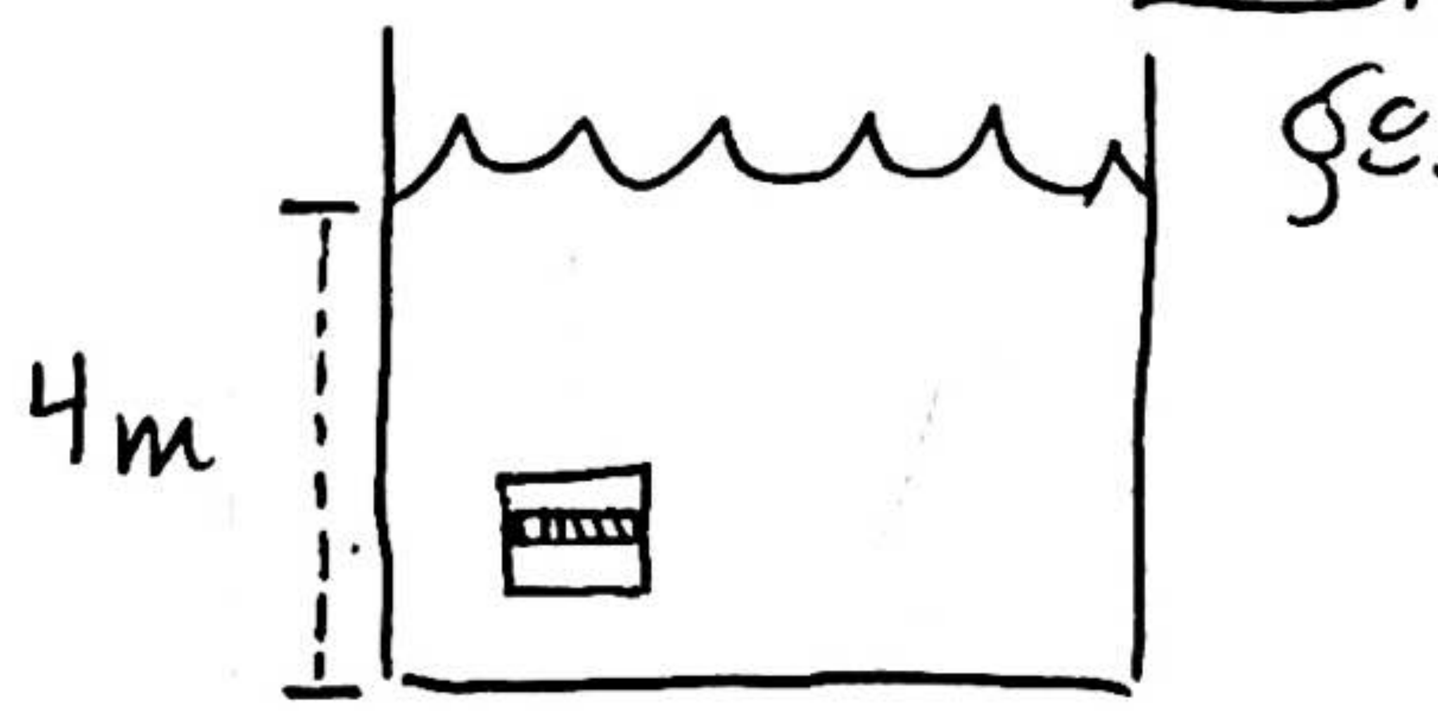
983

9. A large tank is designed with ends in the shape of the region between the curves $y=\frac{x^2}{2}$ and $y=12$, measured in feet. Find the hydrostatic force on one end of the tank if the tank is filled with gasoline to a depth of 8 ft. Note gasoline has weight density $\delta = 42\ {}^{lb}\!/_{ft^3}$.

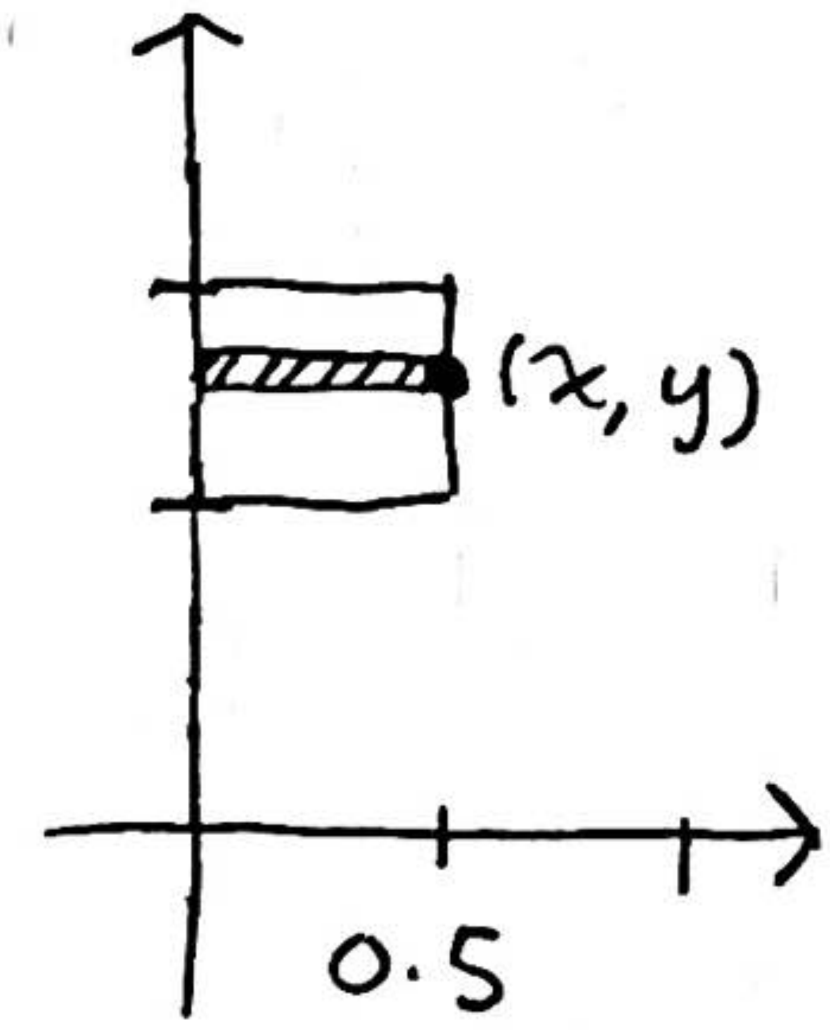
System	Force ×	Distance =	Work
International System of Units (SI)	Newton $\left(N=\frac{kg\cdot m}{s^2}\right)$	Meter (m)	Joule (J)
Centimeter-Gram-Second (CGS)	Dyne (dyn)	Centimeter (cm)	erg
US Customary System or British Engineering	Pound (lb)	Foot (ft)	Foot-pound $(ft\cdot lb)$

Conversion Factors:	
$1N=10^5 dyn \approx 0.225 lb$ $1m \approx 3.28 ft$	$1\ lb \approx 4.45N$
$1J=10^7 erg \approx 0.7338\ ft\cdot lb$	$1\ ft\cdot lb \approx 1.6J = 1.36\times 10^7 erg$

$\boxed{E2}$



$$dF = \rho g(4-y)\left(\tfrac{1}{2}\right) dy$$



$$\int_1^4 \frac{\rho g}{2}(4-y)\,dy$$

$$= 4900\int_1^4 (4-y)\,dy$$

$$= 4900\left[4y - \tfrac{1}{2}y^2\right]\Big|_1^4$$

$$= 4900\left[4(4-1) - \tfrac{1}{2}(4^2 - 1^2)\right]$$

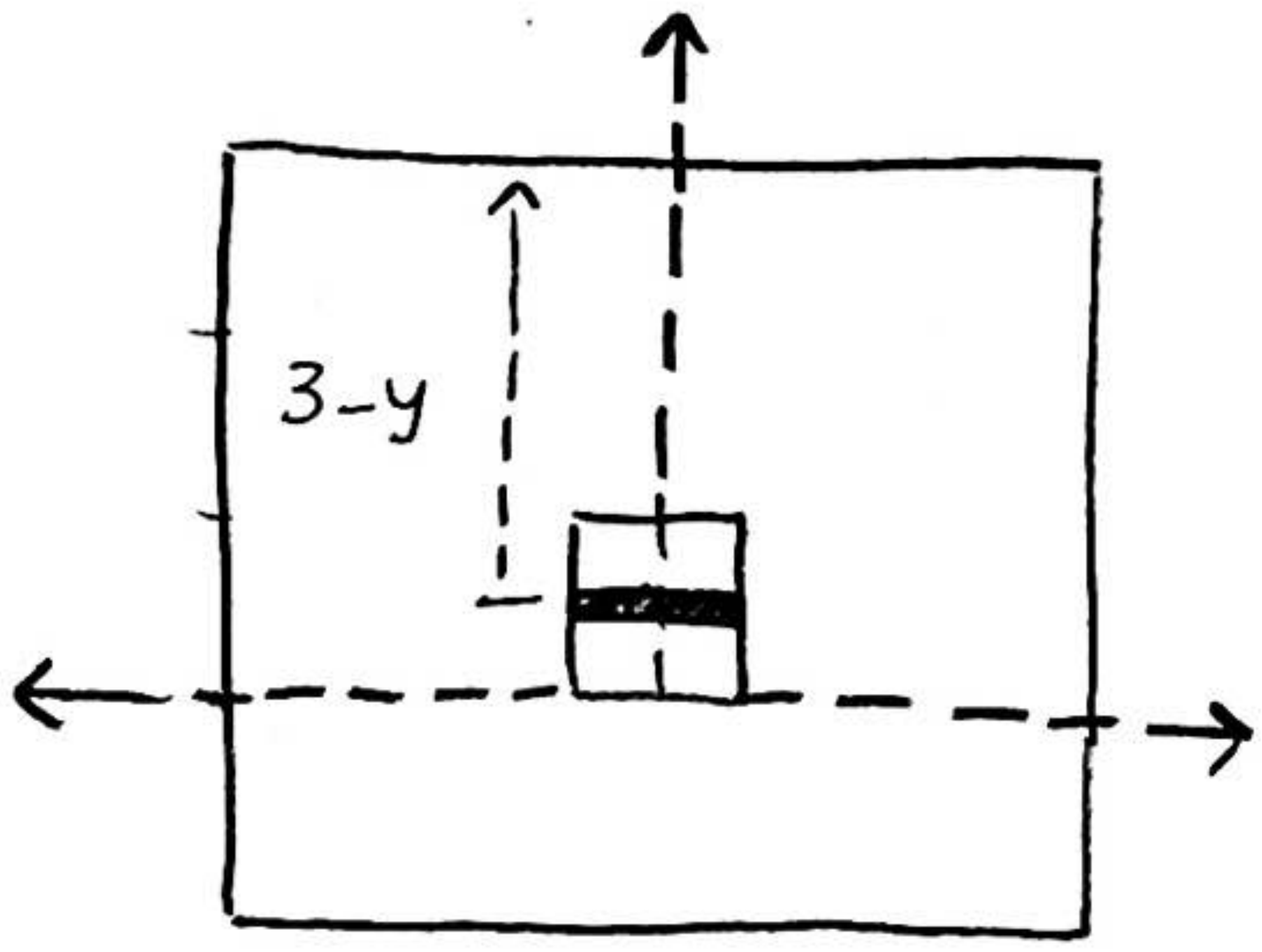
$$= 4900\left[12 - \tfrac{1}{2}(15)\right]$$

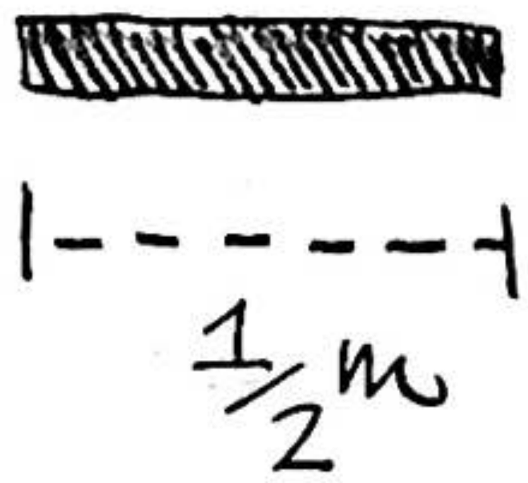
$$= 4900\left[\tfrac{9}{2}\right]$$

$$= 2450(9)$$

$$= 22{,}050 \quad N$$

A DIVING POOL THAT IS 4m DEEP & FULL OF WATER HAS A VIEWING WINDOW ON ONE OF IT'S WALLS. FIND THE FORCE ON A WINDOW THAT IS SQUARE, $\frac{1}{2}$m on a SIDE, W/ THE LOWER EDGE OF THE WINDOW 1M FROM THE BOTTOM OF THE POOL.



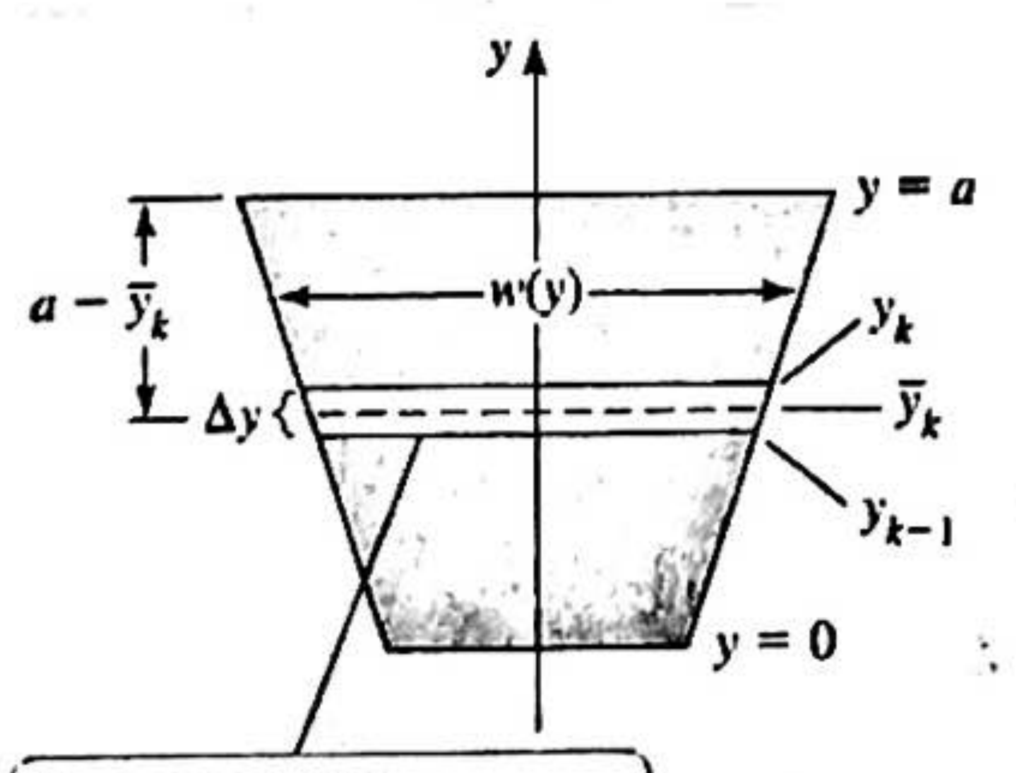


$F = PA$

$= \rho g h (\frac{1}{2}) \Delta y$

$dF = 9800(3-y)\frac{1}{2}dy$

$\int_0^3 \frac{9800}{2}(3-y)\,dy$



Pressure on strip
≈ $\rho g(a - \bar{y}_k)$
Force on strip
≈ $\rho g(a - \bar{y}_k)$ · area of strip
≈ $\rho g(a - \bar{y}_k)\, w(\bar{y}_k)\, \Delta y$

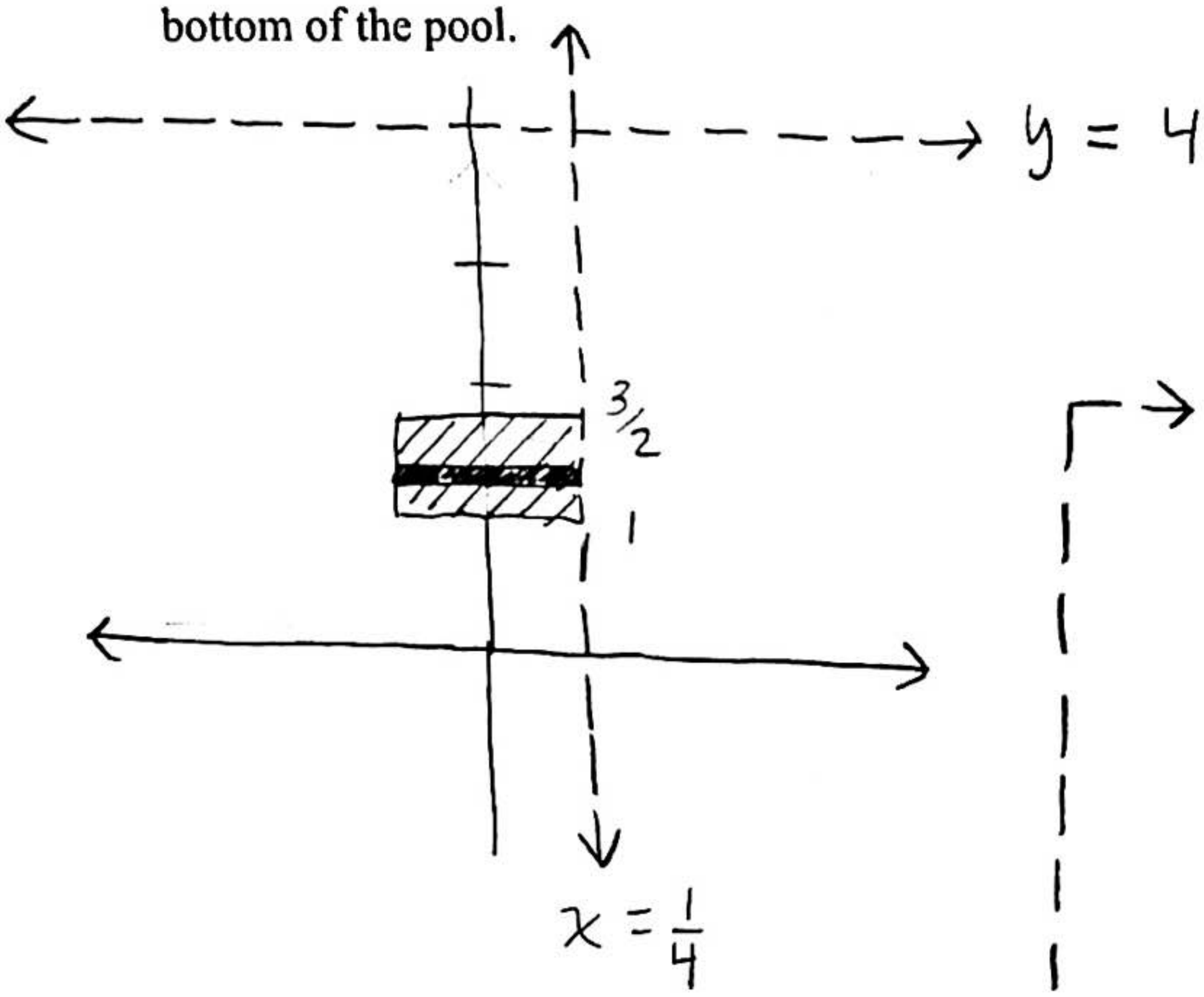
$$F_k = \underbrace{w(\bar{y}_k)\,\Delta y}_{\text{area of strip}}\,\underbrace{\rho g(a - \bar{y}_k)}_{\text{pressure}}.$$

$$F \approx \sum_{k=1}^{n} F_k = \sum_{k=1}^{n} \rho g(a - \bar{y}_k)\, w(\bar{y}_k)\,\Delta y.$$

$$F = \lim_{n\to\infty} \sum_{k=1}^{n} \rho g(a - \bar{y}_k)\, w(\bar{y}_k)\,\Delta y = \int_0^a \rho g(a - y)\, w(y)\, dy.$$

$A = \left(\frac{1}{2}\right)^2$
$= \frac{1}{4}$

EX3: A diving pool that is 4 m deep and full of water has a viewing window on one of its walls. Find the force on a window that is a square, $\frac{1}{2}$ m on a side, with the lower edge of the window 1 m from the bottom of the pool.



$P = \frac{F}{A}$

$F = PA$
$= \rho g h \left(\frac{1}{2}\Delta y\right)$

$F_i = P_i A_i$
$= \rho g (4 - y_i)\left(\frac{1}{2}\Delta y\right)$

$dF = 9800(4-y)\left(\frac{1}{2}\right)dy$

$\int_1^{\frac{3}{2}} \frac{9800}{2}(4-y)\,dy$

$= 4900\int_1^{\frac{3}{2}}(4-y)\,dy$

$= 4900\left[4y - \frac{1}{2}y^2\right]_1^{\frac{3}{2}}$

$= 4900\left[4\left(\frac{3}{2}\right) - \frac{1}{2}\left(\frac{3}{2}\right)^2 - 4 + \frac{1}{2}(1)^2\right]$

$= 4900\left[6 - \frac{9}{8} - 4 + \frac{4}{8}\right]$

$= 4900\left[2 - \frac{5}{8}\right]$

$= \frac{4900}{1}\cdot\left(\frac{11}{16}\right)$

$= 6.7375 \times 10^3\ N$

✓ $F = 12000\rho g$ *Newtons*

✓ $F = 1.176\times10^8$ *Newtons*

$F = \frac{11}{16}\rho g$ *Newtons* ✓

$F = 6.7375\times10^3$ *Newtons* ✓

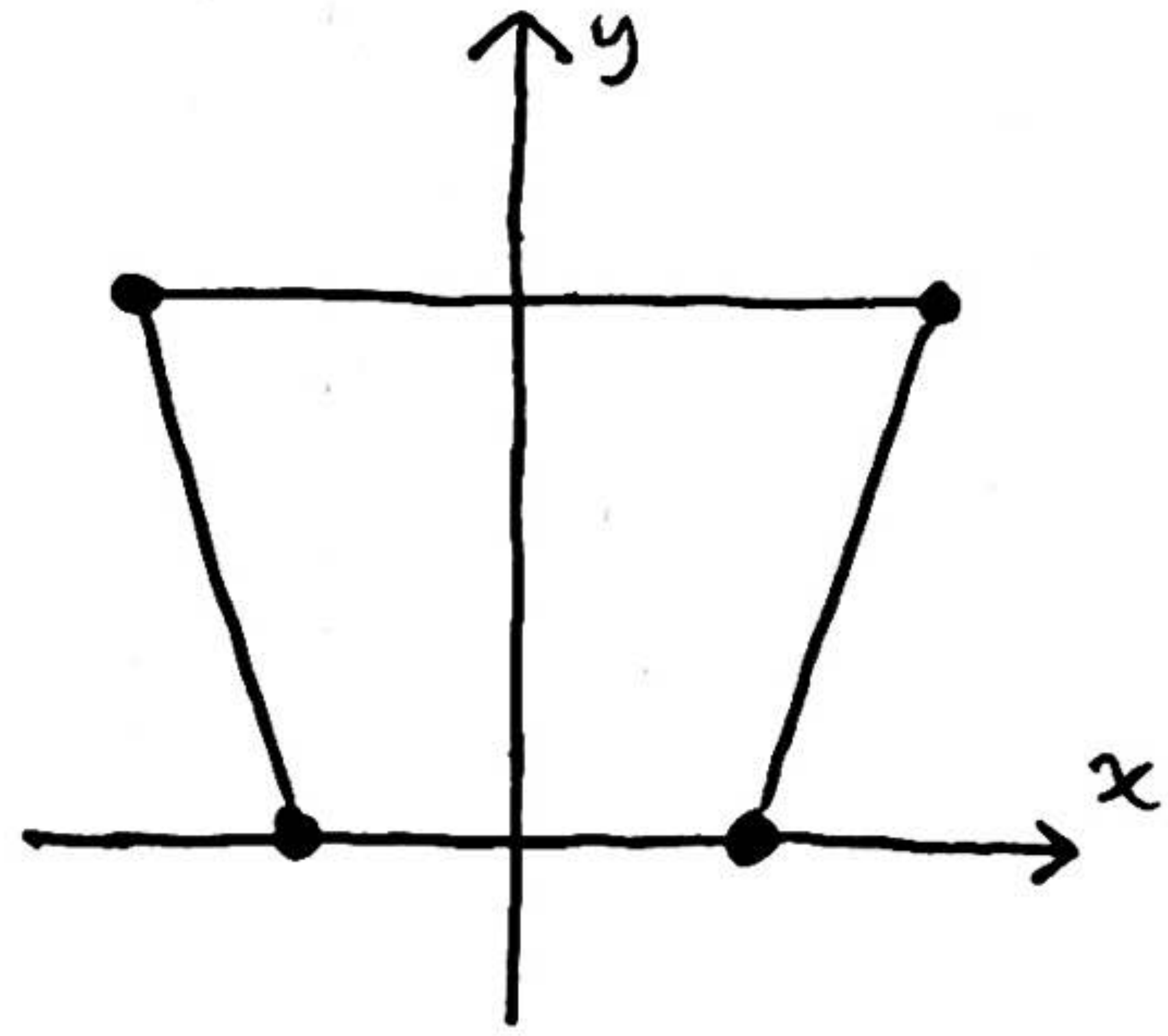
EXAMPLE 2

PRESSURE ON A DAM.

§8.3
EXAMPLE

A LARGE VERTICAL DAM IN THE SHAPE OF A SYMMETRIC TRAPEZOID HAS A HEIGHT OF 30 M, WIDTH OF 20 M @ IT'S BASE & A WIDTH OF 40 M @ IT'S TOP. WHAT IS THE TOTAL FORCE ON THE FACE OF THE DAM, WHEN FULL?

$$F = \int_0^a \rho g(a-y)\,w(y)\,dy$$



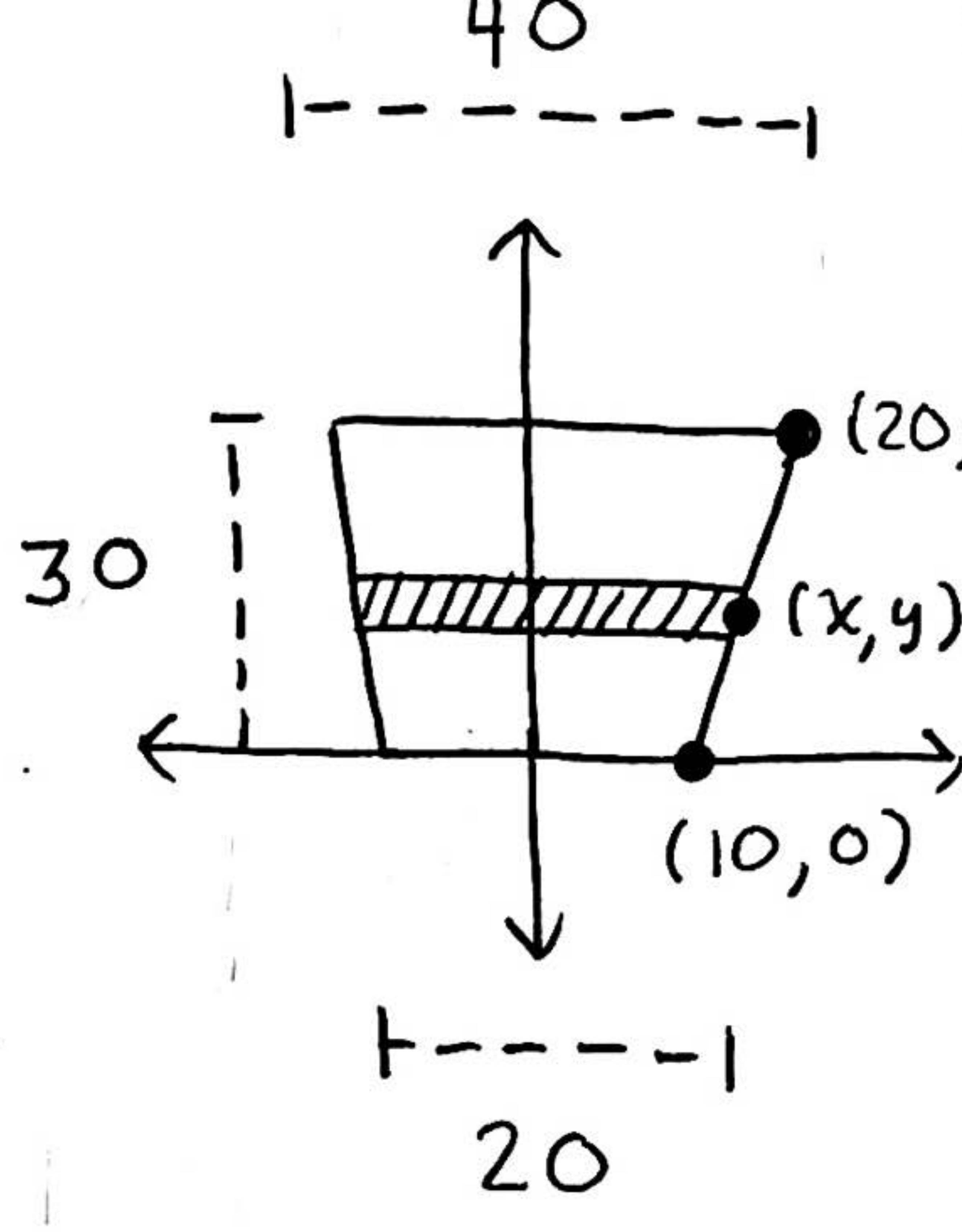
$a =$

$a - y =$

$w(y) = 2(\qquad)$

E1

§8.3



$$F_i = P_i A_i$$

$$= [\rho g(a - y_i)][w(y_i)\Delta y]$$

$$= \rho g(30 - y)[2x\Delta y]$$

$$dF = 2\rho g(30 - y)\left(\frac{y}{3} + 10\right)dy$$

$$F = \int_0^{30} \frac{2}{3}\rho g(30 - y)(y + 30)\,dy$$

$$y - 0 = \frac{30}{10}(x - 10)$$

$$y = 3(x - 10)$$

$$\frac{y}{3} + 10 = x$$

$$= \frac{2}{3}\rho g \int_0^{30} (900 - y^2)\,dy$$

$$= \frac{2}{3}\rho g\left[900y - \frac{1}{3}y^3\right]\Big|_0^{30}$$

$$= \frac{2}{3}\rho g\left[900(30) - \frac{1}{3}(30)^3\right]$$

$$= \frac{2}{3}\rho g\left[(30)^3 - \frac{1}{3}(30)^3\right]$$

$$= \frac{4}{9}\rho g(30)^3$$

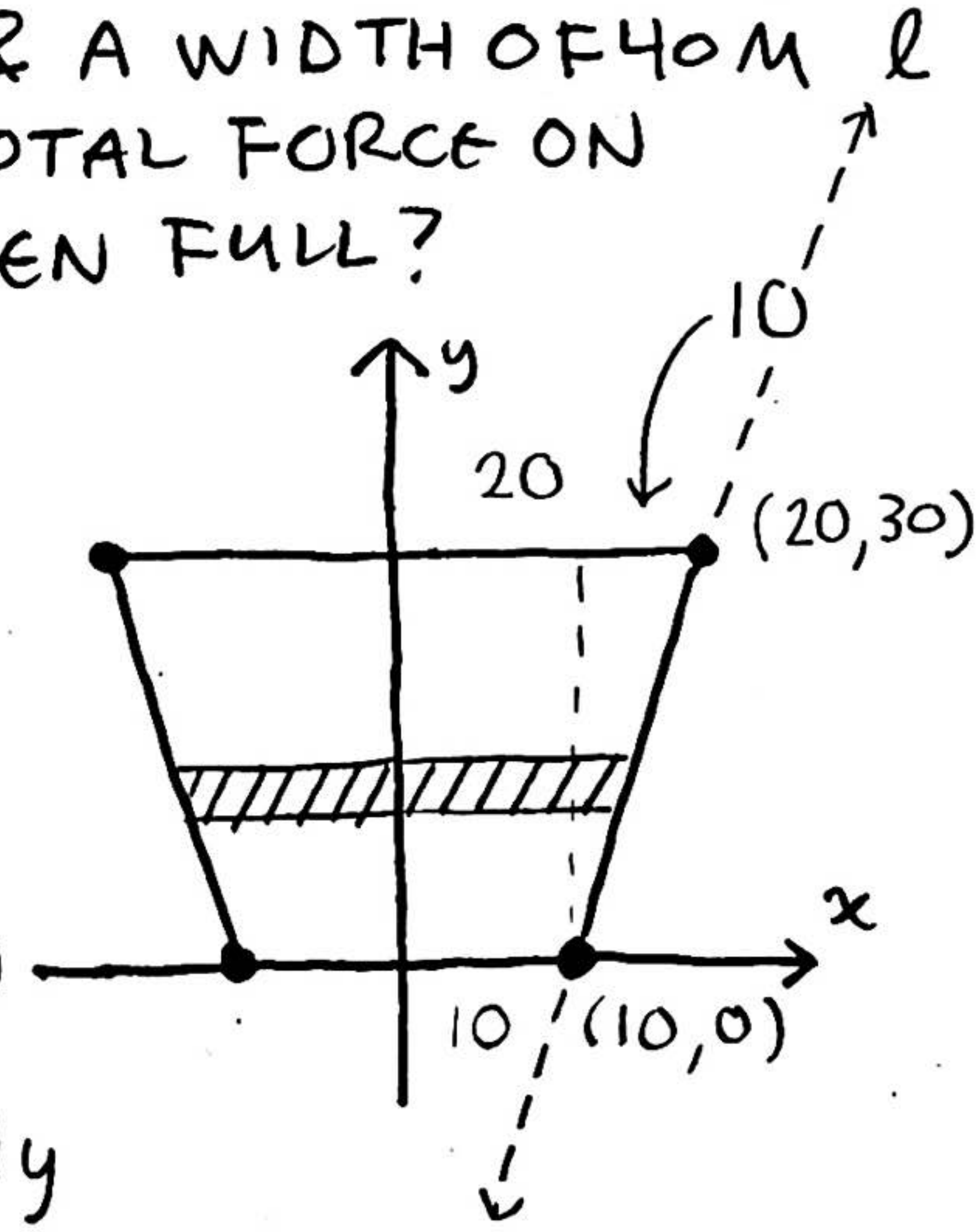
$$= 4\rho g(3)(10^3)$$

$$= 12{,}000\,\rho g = 9{,}800(12{,}000)$$

$$= 1.176 \times 10^8\ \text{N}.$$

PRESSURE ON A DAM.

A LARGE VERTICAL DAM IN THE SHAPE OF A SYMMETRIC TRAPEZOID HAS A HEIGHT OF 30 M, WIDTH OF 20 M @ IT'S BASE & A WIDTH OF 40 M @ IT'S TOP. WHAT IS THE TOTAL FORCE ON THE FACE OF THE DAM, WHEN FULL?



$$F = \int_0^a \rho g (a-y)\, w(y)\, dy$$

$\rho = 1000 \qquad g = 9.8$

$$F = 9800 \int_0^{30} (30-y)\left(\tfrac{2}{3}y+20\right) dy$$

$$F = 9800 \int_0^{30} 20y + 600 - \tfrac{2}{3}y^2 - 20y)\, dy$$

$$= 9800 \int_0^{30} -\tfrac{2}{3}y^2 + 600\, dy$$

$$= 9800 \left[-\tfrac{2}{9}y^3 + 600y\right]_0^{30}$$

$= 9800\,[$[illegible]$]$

$= 9800\,[12000]$

$= 117{,}600{,}000$

$$\boxed{= 1.176 \times 10^8 \text{ N}}$$

$a = 30$

$a - y = 30 - y$

$w(y) = 2\left(\tfrac{1}{3}y + 10\right)$

$\ell : (10,0)\ \&\ (20,30)$

$m = \dfrac{30}{10} = 3$

$y - 0 = 3(x - 10)$

$\tfrac{1}{3}y + 10 = x$

WHAT ABOUT SQUARE WINDOW @ THE BASE OF THE DAM

§8.3

EXAMPLE 1

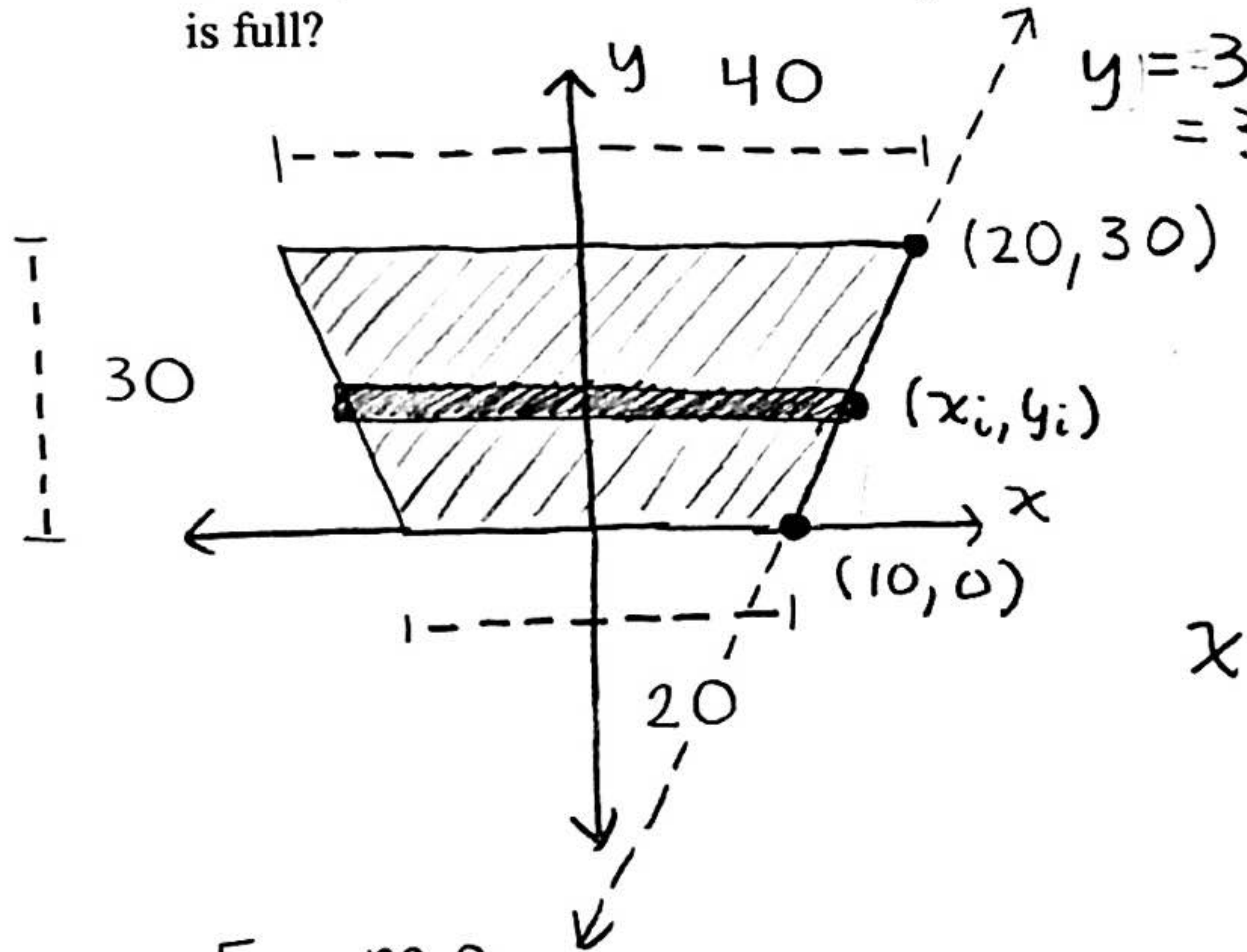
Solving Force/Pressure Problems

1. Draw a y-axis on the face of the dam in the vertical direction and choose a convenient origin (often taken to be the base of the dam).
2. Find the width function $w(y)$ for each value of y on the face of the dam.
3. If the base of the dam is at $y = 0$ and the top of the dam is at $y = a$, then the total force on the dam is

$$F = \int_0^a \rho g \underbrace{(a - y)}_{\text{depth}} \underbrace{w(y)}_{\text{width}} \, dy.$$

Be aware that your origin may change depending on the situation.

EX2: A large vertical dam in the shape of a symmetric trapezoid has a height of 30m, a width of 20m at its base, and a width of 40m at the top. What is the total force on the face of the dam when the reservoir is full?



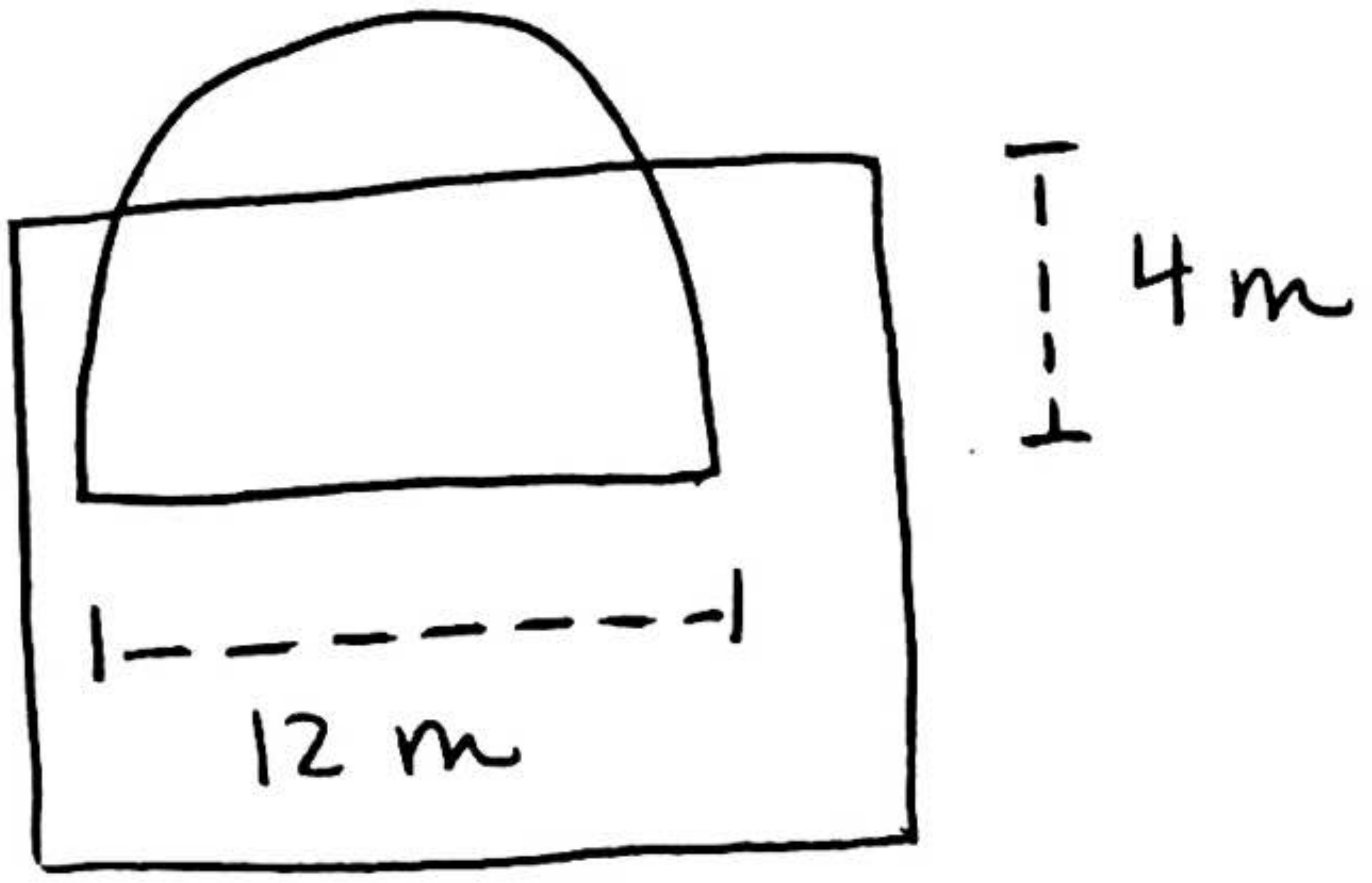
$$y = 3(x-10)$$
$$= 3x - 30$$

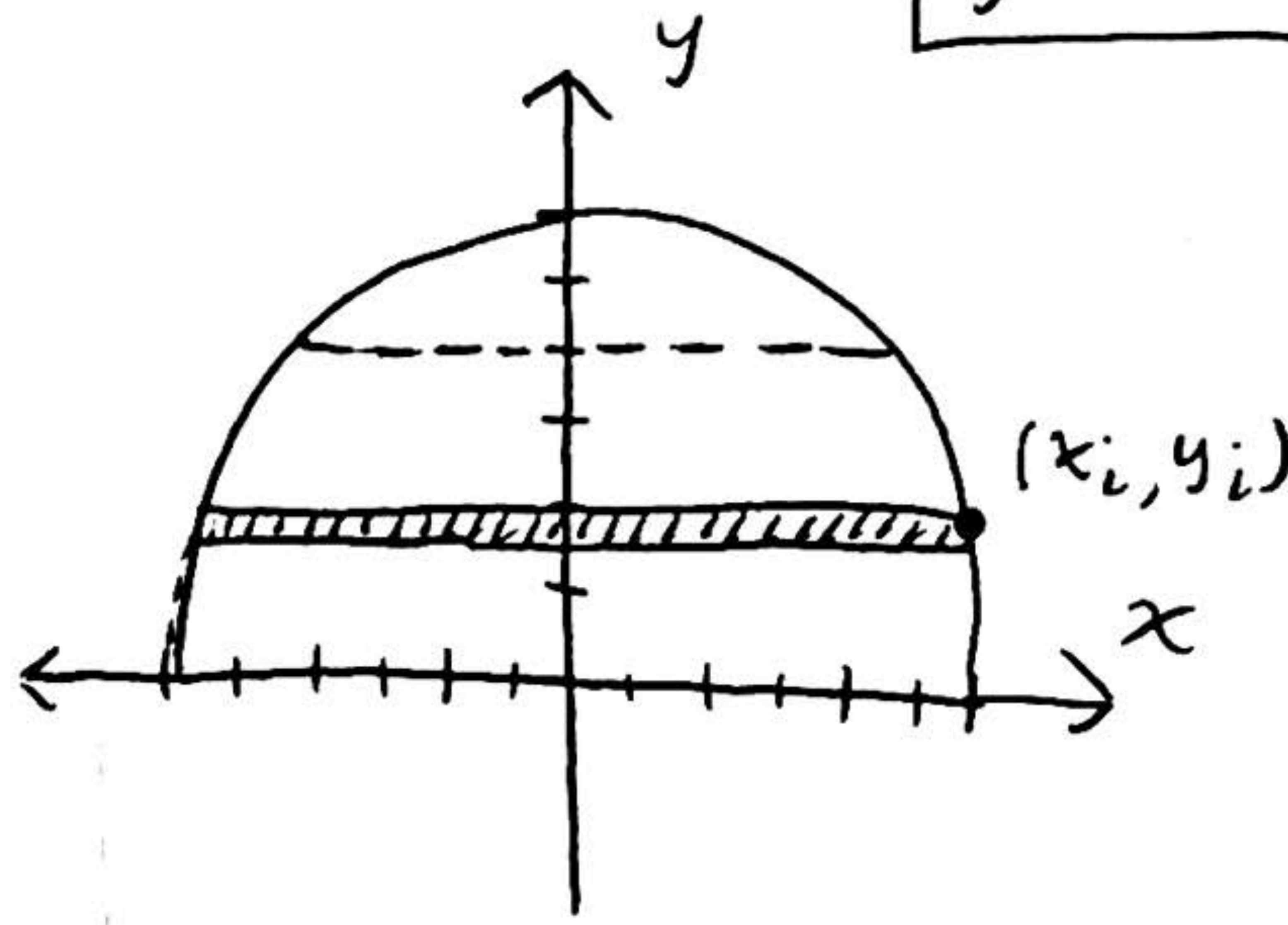
$$x = \frac{1}{3}y - 10$$

$$F = ma$$
$$= \rho g d$$

$$F_i = P_i A_i$$
$$\approx \rho g (a - y_i)(w(y_i)) \Delta y$$
$$dF = \rho g (30 - y)(2 \cdot x) dy$$
$$dF = \rho g (30 - y)(2(\tfrac{1}{3}y - 10)) dy$$
$$F = \int_0^{30} \rho g (30 - y)(\tfrac{2}{3}y - 20) \, dy$$

§8.3.6





$$x^2 + y^2 = 36$$
$$x = \sqrt{36 - y^2}$$

$$P = \frac{F}{A} = \rho d h$$

$$F = ma$$
$$F = \rho V g$$
$$F = \rho A h g$$

$$P = \frac{F}{A}$$
$$= \frac{\rho A h g}{A}$$
$$= \rho h g$$

$$F = PA$$
$$= \rho h g A$$

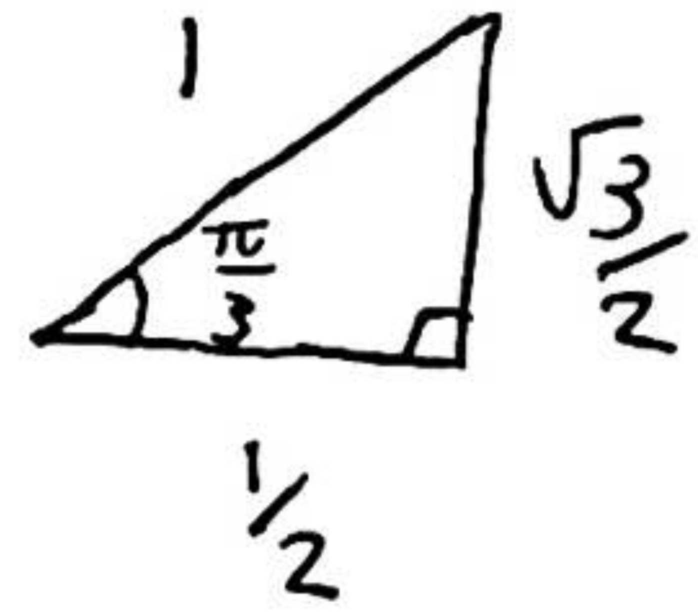
$$F_i = \rho g (4 - y_i)(2x_i)\Delta y$$
$$= 2\rho g (4 - y_i)\sqrt{36 - y_i^2}\,\Delta y$$
$$dF = 2\rho g (4 - y)\sqrt{36 - y^2}\,dy$$

$$F = \int_0^4 2\rho g (4 - y)\sqrt{36 - y^2}\,dy$$
$$= 2\rho g \int_0^4 (4 - y)\sqrt{36 - y^2}\,dy$$

$$\approx 9.04 \times 10^5 \text{ N}$$

§8.3
EXAMPLE 4

$$y' \tan x = a + y \qquad y(\pi/3) = a, \quad 0 < x < \frac{\pi}{2}$$



$$\frac{1}{a+y} \cdot \frac{dy}{dx} = \cot x$$

$$\int \frac{1}{a+y}\, dy = \int \cot x \, dx$$

$$\ln|a+y| = \ln|\sin(x)| + C$$

$$|a+y| = e^{C} \cdot |\sin(x)|$$

$$|a+y| = e^{C} \cdot \sin(x)$$

$$a+y = \pm e^{c} \cdot \sin(x)$$

$$y = -a \pm e^{c} \sin(x)$$

$$a = -a \pm e^{c} \sin\left(\frac{\pi}{3}\right)$$

$$2a = e^{c}\left(\frac{\sqrt{3}}{2}\right)$$

$$\frac{4a}{\sqrt{3}} = e^{c} \qquad \therefore \ c = \ln\left(\frac{4a\sqrt{3}}{3}\right)$$

$$\boxed{y = -a \pm \frac{4a\sqrt{3}}{3} \sin(x)}$$

EXAMPLE 4

$\boxed{99.3}$

$\boxed{\text{CH9}}$

$$x \ln x = y\left(1+\sqrt{3+y^2}\right)y' \quad ; \quad y(1)=1$$

EXAMPLE 5

$\boxed{99.3}$

LET $S(t)$ BE THE TOTAL AMOUNT OF SALT IN THE TANK.

$$S(0) = 20\,kg$$

$$\lim_{t \to \infty} S(t) = 150\,kg$$

$$V = 5000\ L$$

$$\frac{dS}{dt} = S_{IN} - S_{OUT}$$

$$= \left(0.03\,\frac{kg}{L}\right)\left(\frac{25\ L}{MIN}\right) - \frac{S(t)\,kg}{5000\,L}\left(\frac{25\,L}{MIN}\right)$$

$$= \frac{0.75\ kg}{MIN} - \frac{1}{200}\ \frac{S(t)\ kg}{MIN}$$

$$= \frac{3}{4} - \frac{S(t)}{200}$$

$$\frac{dS}{dt} = \frac{3}{4} - \frac{S}{200} = \frac{150 - S}{200}$$

$$200 \frac{ds}{dt} = 150 - s$$

$$\frac{1}{150-s} ds = \frac{1}{200} dt$$

$$\int \frac{1}{150-s} ds = \int \frac{1}{200} dt$$

$$\boxed{c_3 = c_2 - c_1}$$

$$-\ln|150-s| + c_1 = \frac{1}{200} t + c_2$$

$$\boxed{c_4 = -c_3}$$

$$-\ln|150-s| = \frac{1}{200} t + c_3$$

$$\ln|150-s| = \frac{-t}{200} + c_4$$

$$150 - s = e^{-\frac{t}{200} + c_4}$$

$$s = 150 - e^{c_4} e^{-\frac{t}{200}}$$

$$s = 150 - c_4 e^{-t/200}$$

$$s(0) = 20 = 150 - c_4 e^{0}$$

$$20 = 150 - c_4$$

$$c_4 = 130$$

$$\therefore \quad s(t) = 150 - 130 e^{-\frac{1}{200} t}$$

$$S(30) = 150 - 130e^{-\frac{30}{200}}$$

$$= 150 - 130e^{-\frac{3}{20}}$$

$$\approx 38.1$$

WHAT IS $\lim_{t \to \infty} \left(150 - 130e^{-\frac{t}{200}}\right)$?

$\boxed{\text{VERSION 2}}$

EXAMPLE 6

$$S(t) = S_{IN}(t) - S_{OUT}(t)$$

$$\frac{\partial S}{\partial t} = \frac{\partial S_{IN}}{\partial t}(t) - \frac{\partial S_{OUT}}{\partial t}(t)$$

$$\frac{\partial s}{\partial t} = 25(0.03) - \frac{25\,S(t)}{5000}$$

$$\frac{\partial s}{\partial t} = 0.75 - \frac{S(t)}{200}$$

$$\frac{\partial s}{\partial t} = \frac{3}{4}\cdot\frac{50}{50} - \frac{S}{200}$$

$$\frac{\partial s}{\partial t} = \frac{150 - S}{200}$$

$$\frac{1}{150-S}\,\partial s = \frac{1}{200}\,\partial t$$

$$\int \frac{1}{150-S}\,\partial s = \int \frac{1}{200}\,\partial t$$

$\boxed{\S 9.3}$

$$-\ln|150-s| = \frac{1}{200}t + C$$

$$\ln|150-s| = \frac{-1}{200}t - C$$

$$150 - s = e^{\frac{-t}{200} - c}$$

$$150 - s = e^{-c} e^{-\frac{t}{200}}$$

$$S = -e^{-c} e^{-t/200} + 150$$

$$S(0) = -e^{-c} e^{0} + 150$$

$$20 = -e^{-c} + 150$$

$$-130 = -e^{-c}$$

$$e^{-c} = 130$$

$$-c = \ln 130$$

$$c = -\ln 130$$

$$S = -130e^{-\frac{t}{200}} + 150$$

$$t = 30$$

$$S(30) = -130e^{-\frac{30}{200}} + 150$$

$$= -130e^{-\frac{3}{20}} + 150$$

$$\approx 38.10796306\ldots$$

$$= 38.1$$

- -

$$\frac{dp}{dt} = kP$$

$$\frac{1}{P}\,dP = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| + c_1 = kt + c_2$$

$$\ln|P| = kt + c_3$$

$$e^{\ln P} = e^{c_3} e^{kt}$$

$$P = c_4 e^{kt}$$

§9.4

$$\frac{dP}{dt} = kP\left[1 - \frac{P}{M}\right]$$

% POPULATION NOT THERE

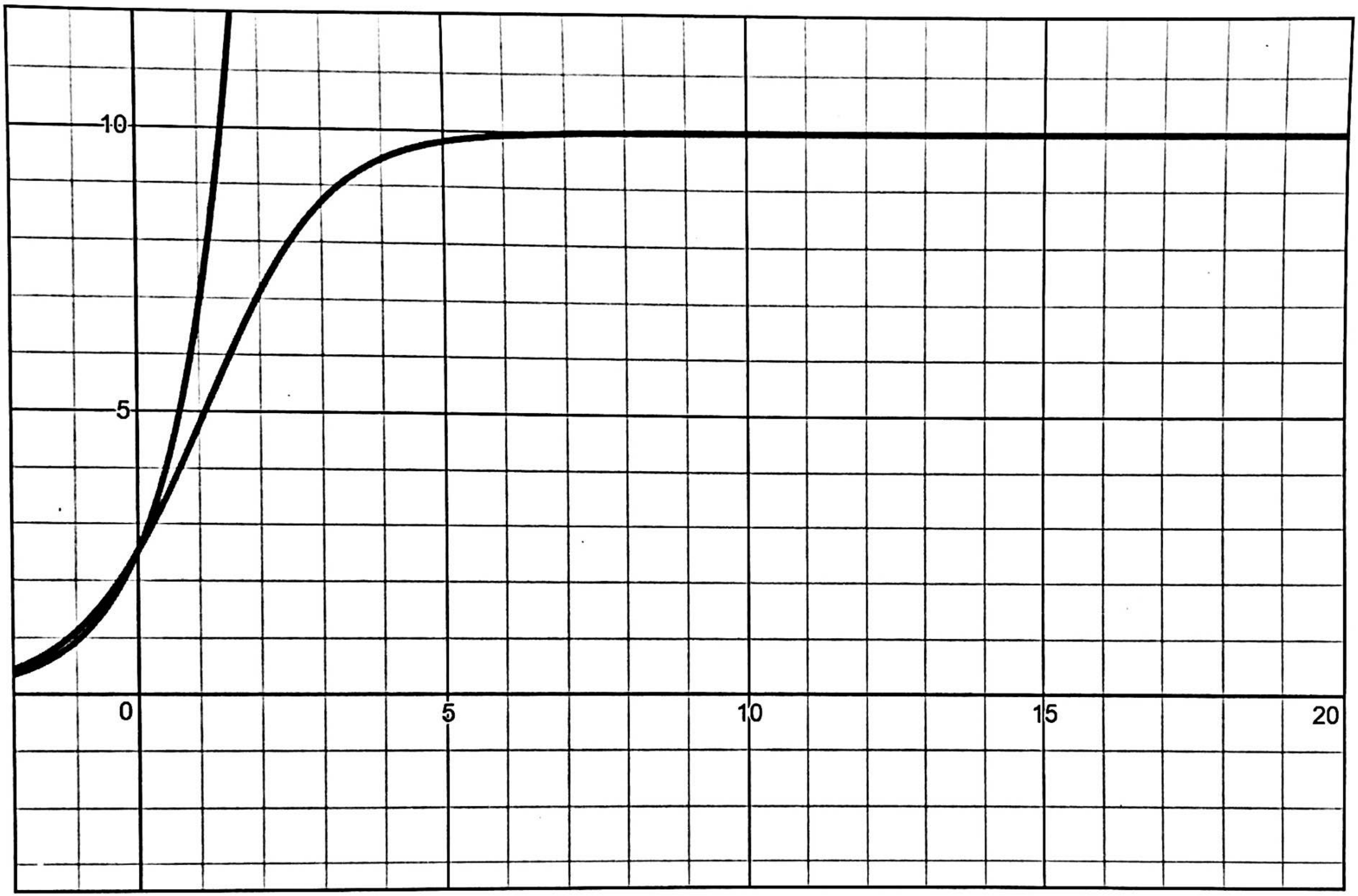
---

§9.4

IF M IS THE MAX POPULATION

THEN $\frac{P}{M}$ IS THE % OF THE MAX

& $1 - \frac{P}{M}$ IS



EXPONENTIAL VS. LOGISTIC

§9.4

$P(0) = 10 \qquad M = 1000$

$P(3) = 15$

IF $\dfrac{dP}{dt} = kP$ THEN

$$\frac{1}{P}\,dP = k\,dt$$

$$\ln P + C_1 = kt + C_2$$

$$\ln P = kt + C_3 \qquad C_3 = C_2 - C_1$$

$$P = e^{kt}e^{C_3}$$

$$P = Ce^{kt} \qquad C = e^{C_3}$$

$P(0) = Ce^{k(0)} = C$

$\therefore \; C = 10$

$P(t) = 10e^{kt}$

$P(3) = 10e^{3k} = 15$

$\dfrac{15}{10} = e^{3k}$

$\dfrac{3}{2} = e^{3k}$

$3k = \ln\left(\frac{3}{2}\right)$

$k = \frac{1}{3}\ln\left(\frac{3}{2}\right)$

$\therefore \; P(t) = 10e^{(\ln\frac{3}{2})t}$

$$P = \frac{M}{1 + Ae^{-kt}}$$

$M = 1000$

$P(0) = 10$

$K = \ln\left(\frac{3}{2}\right)$

$$A = \frac{M - P(0)}{P(0)} = \frac{990}{10} = 99$$

$$P(t) = \frac{1000}{1 + 99e^{-(\ln\frac{3}{2})t}} \qquad \frac{1000}{1 + 99e^{(\ln\frac{2}{3})t}}$$

$$P(t) = 10e^{(\ln\frac{3}{2})t}$$

WHEN DOES EACH MODEL SAY $P = 900$?

$\boxed{\S 9.4}$

$$P(t) = 10e^{(\ln\frac{3}{2})t}$$

$$900 = 10\,e^{(\ln\frac{3}{2})t}$$

$$90 = e^{(\ln\frac{3}{2})t}$$

$$\ln(90) = \ln\left(\tfrac{3}{2}\right) t$$

$$t = \frac{\ln 90}{\ln\frac{3}{2}} \approx 11.1 \text{ UNITS OF TIME}$$

$$P(t) = \frac{1000}{1 + 99e^{(\ln\frac{2}{3})t}}$$

$$900\left(1 + 99e^{(\ln\frac{2}{3})t}\right) = 1000$$

$$1 + 99e^{(\ln\frac{2}{3})t} = \frac{10}{9}$$

$$99e^{(\ln\frac{2}{3})t} = \frac{1}{9}$$

$$e^{(\ln\frac{2}{3})t} = \frac{1}{891}$$

$$\left(\ln\tfrac{2}{3}\right)t = -\ln(891)$$

$$t = \frac{-\ln(891)}{-\ln(3/2)} = \frac{\ln(891)}{\ln(3/2)}$$

$$\approx 16.8$$

UNITS OF TIME.

3

NOTES

$$P(t) = P_0 e^{kt}$$

$$P(t) = \frac{M}{1 + Ae^{-kt}}$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right) P$$

$$\frac{dP}{dt} = kP\cos(rt - \phi)$$

OTHER DE MODELS

§9.4

$$\frac{dP}{dt} = k\left(1-\frac{P}{M}\right)(P-m)$$

$$= Mk\,(M-P)(P-m)$$

$$\frac{1}{(M-P)(P-m)}\,dP = Mk\,dt$$

$$\int \frac{A}{M-P} + \frac{B}{P-m}\,dP = \int Mk\,dt$$

$\boxed{\S 9.4}$

$$\frac{dp}{dt} = kP\left(1 - \frac{m}{P} - \frac{P}{M} + \frac{m}{M}\right)$$

$$= kP - km - \frac{k}{M}P^2 + \frac{m}{M}$$

$$= -\frac{k}{M}P^2 + kP - km + \frac{m}{M}$$

$$= -\frac{k}{M}\left[P^2 - MP + mM - km\right]$$

$$= \frac{-k}{M}\left[P^2 - MP + \frac{M^2}{4} - \frac{M^2}{4} + mM - km\right]$$

$$= \frac{-k}{M}\left[P - \frac{M}{2}\right]^2 - \frac{k}{M}\left[-\frac{M^2}{4} + mM - km\right]$$

$\boxed{\S 9.5}$

$$\frac{d}{dx} e^{P(x)}$$

$$= e^{P(x)} \cdot \frac{d}{dx} P(x)$$

$$= e^{P(x)} \cdot P'(x)$$

$$\frac{d}{dx} e^{\int P(x)\,dx}$$

$$= e^{\int P(x)\,dx} \cdot P(x)$$

SOLVE THE FOLDE:

$$y' + 2xy = 2x^3.$$

$$e^{x^2} y' + e^{x^2} 2xy = e^{x^2} 2x^3$$

$$\frac{d}{dx}\left(e^{x^2} y\right) = e^{x^2} 2x^3$$

$$\int \frac{d}{dx}(e^{x^2} y)\,dx = \int e^{x^2} 2x^3\,dx$$

$$e^{x^2} y = \int e^{x^2} 2x^3\,dx$$

$$e^{x^2} y = \int e^{x^2} \cdot x^2 \cdot 2x\,dx$$

$$y e^{x^2} = \int u e^u\,du$$

$$y e^{x^2} = u e^u - e^u + C$$

$$y e^{x^2} = x^2 e^{x^2} - e^{x^2} + C$$

$$y = x^2 - 1 + C e^{-x^2}$$

$$I = e^{\int P(x)\,dx}$$

$$P(x) = 2x$$

$$\int P(x)\,dx = x^2$$

$$I = e^{x^2}$$

$$u = x^2$$

$$du = 2x\,dx$$

DOES IT MATTER IF

$$I = e^{x^2} + C\,?$$

§9.5

EXAMPLE 1

$$y' + 2xy = 2x^3$$

$$y' = 2x^3 - 2xy$$

$$\frac{dy}{dx} = 2x^3 - 2xy \quad (???)$$

$$\frac{dy}{dx} + 2xy = 2x^3$$

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 2x^3e^{x^2}$$

$$\frac{d}{dx}\left[e^{x^2}y\right] = 2xe^{x^2}$$

$$\int \frac{d}{dx}\left[e^{x^2}y\right]dx = \int 2xe^{x^2}\,dx$$

$$e^{x^2}y = \int 2xe^{x^2}\,dx$$

$$y = e^{-x^2}\int 2xe^{x^2}\,dx$$

$\boxed{E2}$

SOLVE THE FOLDE

$$y' + \frac{1}{x\ln x} y = 9x^2$$

$$(\ln x)y' + \frac{1}{x}y = \ln x \cdot 9x^2$$

$$\frac{d}{dx}(y\ln x) = 9x^2 \ln x$$

$$y\ln x = \int 9x^2 \ln x \, dx$$

$$y\ln x = 3x^3 \ln x - \int 3x^2 \, dx$$

$$y\ln x = 3x^3 \ln x - x^3 + C$$

$$y = \frac{3x^3 \ln x - x^3 + C}{\ln x}$$

$$P(x) = \frac{1}{x\ln x}$$

$$\int P(x)\,dx$$

$$= \int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u}$$

$$= \ln u + c$$

$$= \ln(\ln x) + c$$

$c = 0$

$$e^{\int P(x)} = \ln x$$

$u = \ln x \qquad dv = 9x^2$

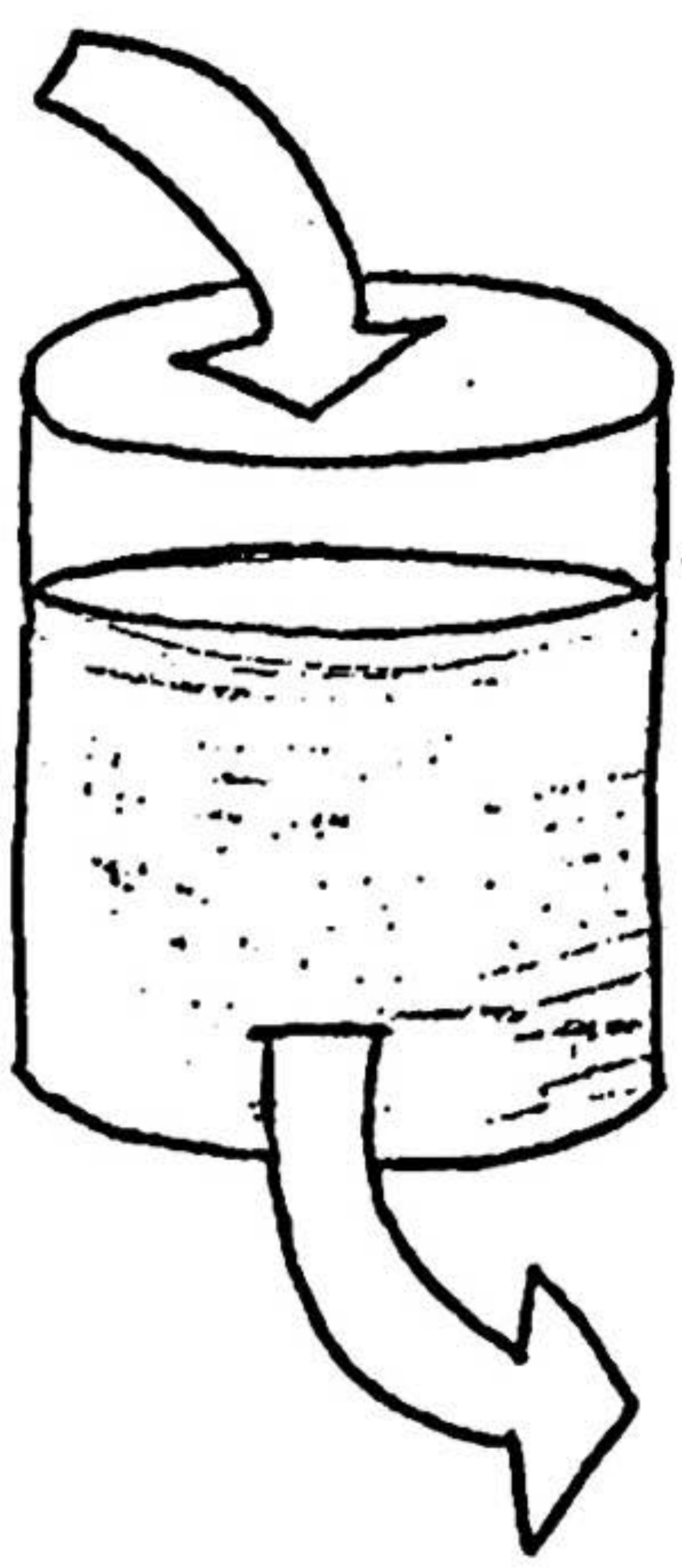
$du = \frac{1}{x}\,dx \qquad v = 3x^3$

$\boxed{\S 9.5}$

A TANK CONTAINS 20 kg OF SALT DISSOLVED IN 5000 L OF WATER.

BRINE THAT CONTAINS 0.01 kg OF SALT PER LITER OF WATER ENTERS THE TANK INITIALLY AT A RATE OF 25L/MIN. THE AMOUNT IS DECREASING BY 1L/MIN W/ THE AMOUNT OF SALT REMAINING CONSTANT. THE SOLUTION IS KEPT THOROUGHLY MIXED AND DRAINS FROM THE TANK AT RATE OF 25 L/MIN. THE TANK HAS A CAPACITY OF 6000L.

HOW MUCH SALT REMAINS IN THE TANK AFTER HALF OF A HOUR. (SHOULD WE BE WORRIED ABOUT ANYTHING?)



LET $S(t)$ REPRESENT THE AMOUNT OF SALT IN THE TANK AT TIME t (MIN).

$$\frac{dS}{dt} = S_{IN} - S_{OUT}$$

$$= \frac{0.01\text{ kg}}{L} \cdot \frac{(25-t)L}{MIN} - \frac{S(t)\text{ kg}}{5000\,L} \cdot \frac{25\,L}{MIN}$$

$$= \frac{1}{10}(25-t) - \frac{S}{200}$$

NEW

TANK CONTAINS 20 kg OF SALT DISSOLVED IN 5000 L OF WATER.

BRINE THAT CONTAINS 0.03 kg OF SALT PER LITER OF WATER ENTERS THE TANK AT A RATE OF 25 L/MIN.

THE TANK IS KEPT THOROUGHLY MIXED BUT DRAINED @ 20 L/MIN.

WHEN WILL THE TANK OVERFLOW IF THE TANK HAS A 6000 L CAPACITY.

$$\frac{dS}{dt}(t) = \left(\frac{0.03\,kg}{1\,L}\right)\left(\frac{25\,L}{MIN}\right) - \frac{S(t)\,kg}{(5000+5t)(L)} \cdot \left(\frac{20\,L}{MIN}\right)$$

$$\frac{ds}{dt} = 0.75 - \frac{4S}{1000+t}$$

$$\frac{ds}{dt} = \frac{3}{4} - \frac{4S}{1000+t}$$

$V(t) = 5000\,L - (25L \quad 20L)t$

$= 5000 + 5t$

$6000 = 5000 + 5t$

$1000 = +5t$

$t = 200$ MIN

$$\frac{ds}{dt} + \frac{4}{1000+t} S = \frac{3}{4}$$

$$\frac{\partial s}{\partial t} + \frac{4}{1000+t} s = \frac{3}{4} \qquad 0 \le t \le 200$$

$$P(t) = 4(1000+t)^{-1}$$

CHOOSE $c = 0$

WHY?

$$\int P(t)\,\partial t = 4\int \frac{1}{1000+t}\,\partial t$$

$$= 4\ln(1000+t) + c$$

$$I = e^{\int P(t)} = e^{4\ln(1000+t)}$$

$$= e^{\ln(1000+t)^4} = (1000+t)^4$$

$$I\frac{\partial s}{\partial t} + I\left(\frac{4}{1000+t}\right)s = \frac{3}{4} I$$

$$(1000+t)^4 \frac{\partial s}{\partial t} + 4(1000+t)^3 s = \frac{3}{4}(1000+t)^4$$

$$\frac{\partial}{\partial t}\left(1000+t\right)^4 s = \frac{3}{4}(1000+t)^4$$

$$(1000+t)^4 s = \int \frac{3}{4}(1000+t)^4\,\partial t$$

SHOULDN'T WE GET A CONSTANT

$$S = \frac{1}{(1000+t)^4}\left(\frac{3}{4}\right)\left[\frac{1}{5}(1000+t)^5 + C_1\right]$$

$$= \frac{3}{4}\left[\frac{1}{5}(1000+t) + C_1(1000+t)^{-4}\right]$$

$$= \frac{3}{20}(1000+t) + C_2(1000+t)^{-4}$$

$$S(0) = 20 = \frac{3}{20}(1000) + C_2(1000)^{-4}$$

$$20 = 150 + C_2(1000)^{-4}$$

$$-130 = C_2(1000)^{-4}$$

$$C_2 = -130(1000)^4$$

$$S(t) = \frac{3}{20}(1000+t) - \frac{130(1000)^4}{(1000+t)^4}$$

$$= \frac{3}{20}(1000+t) - 130\left[\frac{1000}{1000+t}\right]^4$$

$$= \frac{3}{20}(1000+t) - 130\left[1 - \frac{t}{1000+t}\right]^4$$


S
t
(200, 117.307)
-50
0
50
100
150
200
-50
50
100

A TANK CONTAINS 100L OF WATER.

A SOLUTION W/ SALT CONCENTRATION OF 0.4 kg/L IS ADDED AT A RATE OF 5 L/MIN.

THE SOLUTION IS KEPT MIXED & DRAINED AT A RATE OF 3 L/min.

LET $S(t)$ BE THE AMOUNT OF SALT IN THE TANK @ TIME t.

$$S(t) = S_{IN}(t) - S_{OUT}(t)$$

$$\frac{dS}{dt} = \frac{dS_{IN}}{dt} - \frac{dS_{OUT}}{dt}$$

$$= \left[0.4 \frac{kg}{L}\right]\left[5 \frac{L}{MIN}\right] - \left[\frac{S(t)}{100+(5-3)t} \cdot \frac{kg}{L}\right]\left[3 \frac{L}{MIN}\right]$$

$$= 2 - \frac{3S}{100+2t}$$

$$\frac{\partial S}{\partial t} = 2 - \frac{3S}{100+2t}$$

$$\frac{\partial S}{\partial t} + \frac{3S}{100+2t} = 2$$

$$\frac{\partial s}{\partial t} + \frac{3}{100+2t} S = 2$$

$$\frac{\partial s}{\partial t} + \qquad S$$

$$P(x) = \int \frac{3}{100+2t}\, \partial t$$

$$= \frac{3}{2} \int \frac{1}{50+t}\, \partial t$$

$$= \frac{3}{2} \ln|50+t| + C$$

$$e^{\int p(x)}$$

$$= e^{\frac{3}{2}\ln|50+t|+C}$$

$$= e^{C} \cdot e^{\frac{3}{2}\ln|50+t|}$$

$$=$$

OR JUST

$$= t+50$$

$$y' + \frac{1}{\sqrt{x^2-x}}\, y = \frac{2x-1}{\sqrt{x^2-x}}$$

$$\int P(x) = \int \frac{1}{\sqrt{x^2-x}}\, dx$$

$$e^{\int P(t)} = 2\sqrt{x^2-x} + 2x - 1$$

$$\frac{d}{dx}\left(y\left(2\sqrt{x^2-x} + 2x - 1\right)\right) = \frac{2x-1}{\sqrt{x^2-x}}$$

$$y\left[2\sqrt{x^2-x} + 2x - 1\right] = \int \frac{2x-1}{\sqrt{x^2-x}}\, dx$$

$$y = \frac{1}{2\sqrt{x^2-x} + 2x - 1} \int u^{-\frac{1}{2}}\, du$$

$$= \frac{2\sqrt{x^2-x}}{2\sqrt{x^2-x} + 2x - 1} + C$$

$$y' = f(x)\,y$$

$$\frac{dy}{dx} = f(x)\,y$$

$$\frac{1}{y}\,dy = f(x)\,dx$$

$$\int \frac{1}{y}\,dy = \int f(x)\,dx$$

$$\ln|y| + C_1 = F(x) + C_2$$

$$\ln|y| = F(x) + C$$

$$|y| = e^{F(x)}\,e^{C}$$

$$\boxed{y = \pm k\,e^{F(x)}}$$

$$y' = y^2$$

$$\frac{1}{y^2}\,dy = dx$$

$$-\frac{1}{y} = x + c$$

$$\boxed{y = -\frac{1}{x+c}}$$