## §9.1 | Differential Equations

A differential equation is an equation that relates a function to its derivatives. They are some of the hardest equations to solve, many of them being unsolvable. Yet, since they arise in science and nature so often a great deal of effort is spent developing techniques to solve them.

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**Example 1:** Why would it be hard to figure how fast a cup of hot liquid cools to room temperature?

The good news for us is that we will only be considering very simple examples of differential equations as a mini-introduction.

**Example 2:** I am thinking of a function y that…

1. $y = y'$
2. $y' = 5$
3. $y' = 3x^2 + 4$
4. $y'' = -y$
5. $y'' = y$
6. $y' = 5y$
7. $y' = \frac{1}{y}$
8. $y' = y^2$
9. $y' = x + y$

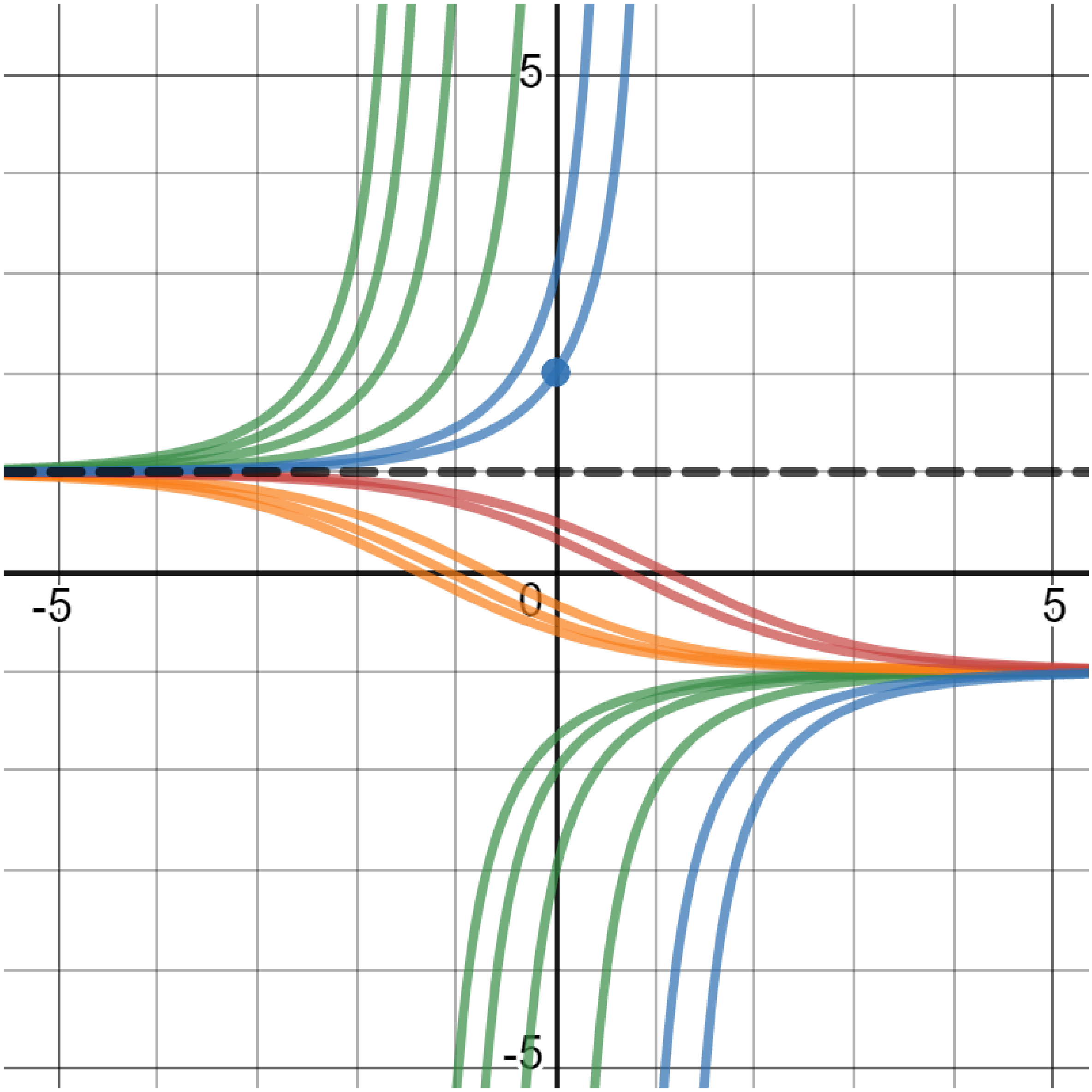
**Example 3:** Verify that every function defined by

$$y = \frac{1+ce^t}{1-ce^t}$$

Is a solution to the differential equation $y' = \frac{1}{2}\left(y^2 - 1\right)$?

What do we need to do to determine this?

**Example 4:** Find one particular solution to $y' = \frac{1}{2}\left(y^2 - 1\right)$ if y(0) = 2.




*(§9.2 skip)*

## §9.3 | Separable Equations

A separable differential equation is one in which the x−variables and y−variables can be "separated" and the equation can easily be solved by integrating both sides of the equation once the variables have been separated.

**Examples 1−5:**

Solve the following differential equations:

| | | |
|---|---|---|
| $\frac{dy}{dx} = \frac{x^2}{y^2}$ | $y' = x^2 y$ | $4\frac{dI}{dt} + 12I = 60$ |
| $y' = y \tan x$ | $x \ln x = y(1+\sqrt{3+y^2})y'$<br>$y(1) = 1$ | |

**Example 4:** A tank contains 20 kg of salt dissolved in 5000 L of water. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt remains in the tank after half of an hour?

A TANK CONTAINS 20 Kg OF SALT DISSOLVED IN 5000 L OF WATER.

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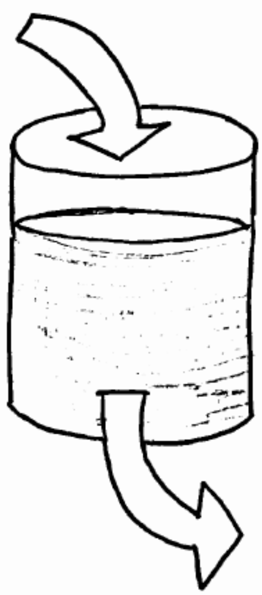
SOL'N



LET $y(t)$ REPRESENT THE AMOUNT OF SALT IN THE TANK AT TIME $t$ (HOURS).

*(Note the typos)*

## §9.4 | Models for Population Growth

One of the amazing relationships differential equations (DE) can model and solve are different models for population growth. We will look at two different models: natural growth and

**Natural growth** is based on the assumption that a population grows proportional to the size of the population

$$\frac{dP}{dt} = kP$$

I think we can solve this equation by guessing.

Considering our real world intuitions, is it reasonable that a population would grow exponentially? Populations usually hit what is called "the carrying capacity" of the environment. If we assume a population as it begins to grow will approximately grow exponentially and then at some point "level off" we get the following equation:

$$\frac{dP}{dt} \approx kP$$
$$\frac{1}{P}\frac{dP}{dt} \approx k$$
$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Where M is the carrying capacity. This model is interesting. Initially it looks exponential, and then it levels of and hits a celling, right? Let's try to solve for P.

What does $1 - \frac{P}{M}$ represent?

## THE LOGISTIC EQUATION

$$\frac{dP}{dt} = kP\left[1 - \frac{P}{M}\right]$$

$1 - \frac{P}{M} \approx$ COMPLIMENT OF % OF M THAT POPULATION REPRESENTS.

$$\frac{1}{P(1-P/M)}\,dP = k\,dt$$

$$\int \frac{A}{P} + \frac{B}{\frac{1}{M}P-1}\,dP = \int k\,dt$$

$1 = A\left(\frac{P}{M}-1\right) + PB$

IF $P=M$

$1 = A\left(\frac{M}{M}-1\right) + MB$

$B = +\frac{1}{M}$

IF $P=0$, $A=-1$

$$\int \frac{-1}{P} + \frac{\frac{1}{M}}{\frac{1}{M}P-1}\,dP = kt + C$$

$$\int \frac{-1}{P} + \frac{1}{P-M}\,dP = kt + C$$

$$\ln|P| + \ln|P-M| = kt + C$$

$P < M$

$0 < M - P$

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln\left|\frac{P}{M-P}\right| = kt + C$$

$$\frac{P}{M-P} = e^{C}e^{kt}$$

$A = e^{C}$ (FOR EASE)

$$P = (M-P)Ae^{kt}$$

$$P(1+Ae^{kt}) = MAe^{kt}$$

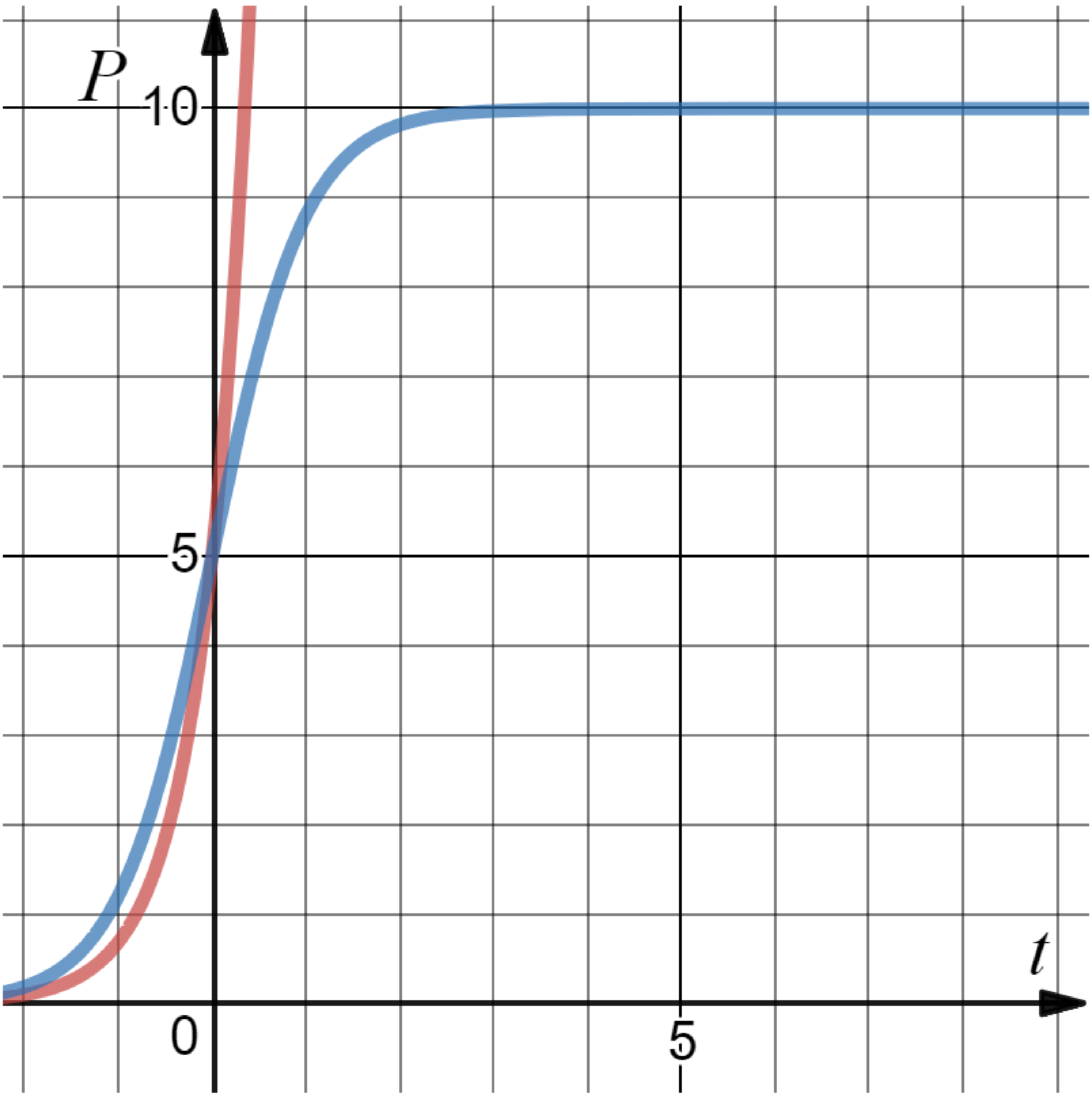
$$P = \frac{MAe^{kt}}{1+Ae^{kt}}$$

$$\boxed{P = \frac{M}{1+Ae^{-kt}}}$$

NOTICE, $P(0) = \frac{M}{1+A}$

$$\left[A = \frac{M-P(0)}{P(0)}\right]$$

$$\lim_{t\to\infty} P(t) = M$$




**Example 1:** Suppose a population has a carrying-capacity of M = 1000 begins with a population of 10 individuals. If there are 15 individuals after 3 units of time, compare the both models of growth we have seen thus far.

## §9.5 | First-Order *Linear* Differential *Equations* (FOLDE)

**Question 1:** What two functions are their own derivative?

**Question 2:** What is $\frac{d}{dx}(e^{P(x)})$? What is $\frac{d}{dx}(e^{\int P(x)dx})$?

*Now for a magic trick….*

**Example 1:** $y' + 2xy = 2x^3$

A **First Order Linear Differential Equations** (FOLDE) is a DE that can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

All **FOLDE** can be solved by multiplying by an **integrating factor**: $I = e^{\int P(x)dx}$.

**Example 2:** $y' + \frac{1}{x\ln x}y = 9x^2$

**Example 3: (new tank) problem**

**Example 3 (ITP):** A particular circuit congaing and electromotive force, a capacitor with a capacitance of C farads (F), and a resister with a resistance of R ohms $(\Omega)$. The voltage drop across the capacitor is $\frac{Q}{C}$ , Where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E$$

$$RI(t) + \frac{Q(t)}{C} = E(t)$$

But, $I(t) = \frac{dQ}{dt}(t)$ (from Physics). If R = 5 $\Omega$, C = 0.05 F and $E(t) = 60\,V$ , find the charge function $Q(t)$ if the initial charge Q(0) = 0 coulombs. Find the current function $I(t)$.

**Solution:**

$$5I(t) + \frac{Q(t)}{0.05} = E(t)$$

$$5\frac{dQ}{dt}(t) + \frac{Q(t)}{0.05} = E(t)$$

$$5\frac{dQ}{dt} + \frac{Q}{1/20} = 60$$

$$5\frac{dQ}{dt} + 20Q = 60$$

$$\frac{dQ}{dt} + 4Q = 12$$

*Answer:* $Q(t) = 3(1 - e^{-4t})$ & $I(t) = 12e^{-4t}$

*(§9.6 skip)*